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APPROXIMATION OF THE BESSEL EIGENVALUE PROBLEM BY FINITE DIFFERENCES*

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Abstract. Difference equation problems are studied whose solutions are estimates of the solutions of the eigenvalue problem

$$Ly \equiv -y'' - \frac{1}{x}y' + \frac{\nu^2}{x^2}y = \lambda y, \qquad x \in (0, 1),$$

$$y(0) = 0, \qquad y(1) = 0,$$

for $0 < \nu < 1$. Three-point difference operators, L_h, are constructed so that $L_h\varphi^{(k)}(x_j) = L\varphi^{(k)}(x_j)$, $k = 1, 2, 3, j = 1, \cdots, N$, where $\{\varphi^{(k)}\}_{k=1}^{3}$ is a given set of functions and $x_j = j/(N + 1)$.

An upper bound on the distance between an eigenvalue, λ, of L and the set $\{\Lambda_j\}_1^N$ of eigenvalues of any three-point difference operator L_h whose coefficients satisfy prescribed conditions is obtained. The bound is a product of a constant and the max norm of the truncation error of L and L_h with respect to the eigenfunctions of L corresponding to λ. Two appropriately constructed difference operators are shown to have coefficients which satisfy the prescribed conditions, and for these, $\min_j|\lambda - \Lambda_j|$ is shown to be $O(h^2)$.

1. Introduction. The eigenvalues of the Sturm–Liouville problem

$$\begin{aligned} &(p(x)y')' - q(x)y' + \lambda r(x)y = 0, \qquad x \in (0, 1), \\ &a_0y(0) - a_1y'(0) = 0, \qquad b_0y(1) + b_1y'(1) = 0, \qquad a_i \geqq 0, \qquad b_i \geqq 0, \\ &|a_0| + |a_1| \neq 0, \qquad |b_0| + |b_1| \neq 0, \end{aligned} \tag{1}$$

are frequently approximated by the eigenvalues of a corresponding finite difference problem, where the finite difference problem is constructed in such a way that the differential operators in (1) are replaced by suitably chosen finite difference operators over an equally spaced mesh of length h. If the coefficients of (1) are sufficiently well-behaved, that is, if $p, r > 0$, $q \geqq 0$, and p, q and r are sufficiently smooth on $[0, 1]$, finite difference approximations can be found such that the eigenvalues of the difference problem are within $O(h^2)$ of the eigenvalues of (1). For example, Keller has shown this to be true if p', q and r are continuous on $[0, 1]$ and all derivatives are replaced by their standard central difference approximations [4, pp. 135–137]. Hubbard [3] has shown that a similar result holds when p, q, r are piecewise continuously differentiable, the coefficients are replaced by averages of the coefficients over the mesh, and each derivative is replaced by its forward difference approximation.

Variational methods yield higher order approximations to the eigenvalues of (1). See, for example, [1]. The assumption that p and r are positive in the closed interval $[0, 1]$ is also made for determination of error bounds of approximations obtained in this way.

Sturm–Liouville problems with singularities arising from p vanishing at one of the endpoints of the interval are not included in the error analyses such as those mentioned above. It might be expected that for such operators the order of the

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error bound might be lower than it is for nonsingular operators. An example of such a problem is the Bessel eigenvalue problem

$$(xy')' - \frac{v^2}{x}y + \lambda xy = 0, \qquad x \in (0,1), \tag{2}$$
$$y(0) = 0, \qquad y(1) = 0,$$

which has a regular singular point at $x = 0$. The results given in Table 1 in § 4 show that the finite difference problem approximating (2) which is defined by replacing each derivative by the standard central difference approximation has smallest eigenvalue which differs from the smallest eigenvalue of (2) by $O(h^{2v})$ for $0 < v < 1$ and $O(h^2)$ for $v \geqq 1$. We see that Keller's result does not extend to problem (2) for $0 < v < 1$.

To study the effect of such a singular point, we limit our study (based on [2]) to approximation of the eigenvalues of (2) by the eigenvalues of the tridiagonal $N \times N$ matrices obtained from the approximations of (2) by the finite difference problem

$$L_h u_j \equiv \alpha_j u_{j-1} + \beta_j u_j + \gamma_j u_{j+1} = \Lambda u_j, \qquad j = 1, \cdots, N, \tag{3}$$
$$u_0 = 0, \qquad u_{N+1} = 0,$$

with constant mesh length $h = 1/(N+1)$, N a positive integer, where L_h is a finite difference approximation to the differential operator L given by

$$Ly \equiv -y'' - \frac{1}{x}y' + \frac{v^2}{x^2}y. \tag{4}$$

Note that (2) is the eigenvalue problem for the operator L with appropriate boundary conditions.

A bound on the distance of any given eigenvalue, λ_j, of (2) from the set of eigenvalues of (3) is given by Theorem 1 as a constant times the max norm of the truncation error of L and L_h with respect to the eigenfunction associated with λ_j when the coefficients of L_h satisfy certain conditions.

We study three difference operators denoted by $L_h^{(1)}$, $L_h^{(2)}$, $L_h^{(3)}$. The first of these is the standard central difference approximation to L on the mesh. We show that the operators $L_h^{(1)}$ and L are inconsistent in the sense that the max norm of the truncation error is unbounded as $h \downarrow 0$ when $0 < v < 1$. These two operators are consistent with respect to a different norm, but this fact is irrelevant to the present study. The truncation error of $L_h^{(1)}$ and L is zero for any quadratic polynomial $v(x) = ax^2 + bx + c$, but quadratic polynomials yield poor approximations to the eigenfunctions of L, which behave like x^v as $x \downarrow 0$ for $0 < v < 1$. This is the reason that the max norm of the truncation error is unbounded. Nevertheless, we observe that the eigenvalues of the difference problem converge to those of the differential problem, but at a slower rate than for the nonsingular Sturm–Liouville problems.

The operator $L_h^{(2)}$ is derived in such a way that account is taken of the behavior of the eigenfunctions of (2) near the singular point. The coefficients α, β, γ are determined such that for any function $v(x) = ax^{v+2} + bx^{v+1} + cx^v$, the value of $L_h^{(2)}v(x)$ is the same as the value of $Lv(x)$ at each mesh point, i.e., the truncation error for $L_h^{(2)}$ and L is zero for such functions. The eigenfunctions of (2) are constant multiples of $J_\nu(\lambda^{1/2}x)$, where λ is the corresponding eigenvalue, and $J_\nu(x)$ is

$$J_\nu(x) = \sum_{k=0}^{\infty} c_{2k}x^{\nu+2k}, \quad c_{2k} = \frac{1}{k!\Gamma(\nu + k + 1)2^{\nu+2k}}.$$

Thus, the operator $L_h^{(2)}$ is constructed so that a better approximation to the eigenfunctions near $x = 0$ is achieved than when quadratic polynomials are used. Furthermore, away from $x = 0$, functions of the form $v(x) = ax^{\nu+2} + bx^{\nu+1} + cx^\nu$ also yield good local approximations to the eigenfunctions. It follows, as it is shown in § 2, that the max norm of the truncation error is $O(h^2)$, and since $L_h^{(2)}$ satisfies the hypotheses of Theorem 1, it is a corollary that the eigenvalues of $L_h^{(2)}$ differ from those of L by at most $O(h^2)$.

The operator xL is self-adjoint, but the operator $xL_h^{(2)}$ is not. To better approximate this property of L, the difference operator $L_h^{(3)}$ is constructed by finding coefficients α, β, γ such that for any function of the form $v(x) = ax^{\nu+2} + bx^\nu$, the value of $L_h^{(3)}v(x)$ is the same as the value of $Lv(x)$ at the mesh points and, in addition, the operator $xL_h^{(3)}$ is self-adjoint. The truncation error of $L_h^{(3)}$ and L is shown to be $O(h^2)$, and it is a corollary of Theorem 1 that the eigenvalues of $L_h^{(3)}$ differ from those of L by at most $O(h^2)$.

In § 4 numerical results are displayed which illustrate these conclusions.

2. Truncation error analysis. The operator $L_h^{(1)}$ constructed so that $L_h^{(1)}v(x) = Lv(x)$ at all mesh points for all quadratic polynomials v is given by

$$(5)\quad L_h^{(1)}u_j = \left(-\frac{1}{h^2} + \frac{1}{2jh^2}\right)u_{j-1} + \left(\frac{2}{h^2} + \frac{\nu^2}{j^2h^2}\right)u_j + \left(-\frac{1}{h^2} - \frac{1}{2jh^2}\right)u_{j+1}, \qquad j = 1, \cdots, N,$$

for any set of numbers $\{u_j\}$. We define the set of functions $\mathscr{F}_\nu$ by

$$(6)\quad \mathscr{F}_\nu = \left\{y \,|\, y(x) = \sum_{h=0}^{\infty} c_{2k}x^{\nu+2k}, \quad y(x)/x^\nu \in C^4[0, 1]\right\}.$$

We note that all eigenfunctions of (2) belong to $\mathscr{F}_\nu$. Application of Taylor's theorem with remainder to any function $y \in \mathscr{F}_\nu$ gives

$$(7)\quad L_h^{(1)}y(x_j) - Ly(x_j) = -\frac{h^2}{24}y^{\mathrm{iv}}(\zeta) + \frac{h^2}{6jh}y'''(x_j) + \frac{h^2}{24jh}[y^{\mathrm{iv}}(\xi) - y^{\mathrm{iv}}(\eta)], \qquad j = 1, \cdots, N,$$

where $x_{j-1} < \eta < \zeta < \xi < x_{j+1}$. As $h \downarrow 0$, then $x_j = jh$, with j fixed, is such that $x_j \downarrow 0$. Hence, for fixed j, $y'''(x_j) = O(h^{\nu-3})$ and $y^{\mathrm{iv}}(x_j) = O(h^{\nu-4})$ as $h \downarrow 0$. Therefore, by (7), we have in general that

$$(8)\quad \|L_h^{(1)}y(x_j) - Ly(x_j)\|_\infty = |\tfrac{1}{24}\nu(\nu - 1)^2(\nu - 2)c_0 j^{\nu-4}h^{\nu-2}| + O(h^\nu), \qquad j = 1, \cdots, N, \quad y \in \mathscr{F}_\nu,$$

where $\|\cdot\|_\infty$ denotes the max norm. Since $0 < \nu < 1$, we have that $L_h^{(1)}$ is not consistent with L with respect to the class of functions $\mathscr{F}_\nu$ and in the max norm.

The operator $L_h^{(2)}$ is constructed so that $L_h^{(2)}v(x) = Lv(x)$ for $x = jh$, $j = 2, \cdots, N$, and for all v of the form $v(x) = ax^{\nu+2} + bx^{\nu+1} + cx^\nu$. Such a construction is accomplished by solving, for each value of j, the system of three equations given by

$$\alpha_j x_{j-1}^{\nu+k} + \beta_j x_j^{\nu+k} + \gamma_j x_{j+1}^{\nu+k} = L x_j^{\nu+k}, \qquad k = 0, 1, 2, \tag{9}$$

for $\alpha_j, \beta_j, \gamma_j$. In the case where $j = 1$, we note that $x_{j-1} = x_0 = 0$, so that (9) reduces to three equations in two unknowns since α_1 has coefficient zero in all three equations. In order to make the system solvable, we eliminate the equation for $k = 1$, reducing the system to two equations. In other words, we make $L_h^{(2)}$ agree exactly with L for all functions of the form $ax^{\nu+2} + bx^\nu$ at the first mesh point. The operator $L_h^{(2)}$ constructed in this way is

$$L_h^{(2)} u_1 \equiv \frac{4\nu + 4}{3h^2}\left[u_1 - \left(\frac{1}{2}\right)^\nu u_2 \right],$$

$$\begin{aligned} L_h^{(2)} u_j \equiv &\left(\frac{j}{j-1}\right)^\nu \frac{1}{h^2}\left(-1 + \frac{\nu}{j} + \frac{1}{2j}\right) u_{j-1} + \frac{2}{h^2} u_j \\ &+ \left(\frac{j}{j+1}\right)^\nu \frac{1}{h^2}\left(-1 - \frac{\nu}{j} - \frac{1}{2j}\right) u_{j+1}, \qquad j = 2, \cdots, N. \end{aligned} \tag{10}$$

For any $y \in \mathscr{F}_\nu$, $x^{-\nu} y(x)$ is in $C^4[0, 1]$. If $x_{j+1}^{-\nu} y(x_{j+1})$ and $x_{j-1}^{-\nu} y(x_{j-1})$ are computed by Taylor's theorem with remainder, and substituted in (10), one obtains

$$L_h^{(2)} y(x_j) = L y(x_j) + O(h^2), \qquad j = 2, \cdots, N. \tag{11}$$

When $j = 1$, we have

$$\begin{aligned} (L_h^{(2)} - L) y(x_1) &= (L_h^{(2)} - L) \sum_{k=0}^{\infty} c_{2k} x_1^{\nu+2k} \\ &= h^{\nu+2} \sum_{k=2}^{\infty} c_{2k}\left[\frac{4\nu + 4}{3}(1 - 4^k) - \nu^2 + (\nu + 2k)^2\right] h^{\nu+2k-4} \\ &= O(h^{\nu+2}). \end{aligned} \tag{12}$$

Therefore, (11) and (12) give

$$\|(L_h^{(2)} - L) y\|_\infty = O(h^2), \qquad y \in \mathscr{F}_\nu. \tag{13}$$

As mentioned in § 1, the differential operator xL defined by

$$xLu(x) = (-xu')' + \frac{\nu^2}{x} u, \qquad 0 < x < 1, \tag{14}$$

is self-adjoint. The finite difference operator $L_h^{(3)}$ is constructed so that it agrees with L on the mesh for functions of the form $ax^{\nu+2} + bx^\nu$, and the operator $xL_h^{(3)}$ defined by

$$(xL_h^{(3)}) u_j = jh L_h^{(3)} u_j, \qquad j = 1, \cdots, N, \tag{15}$$

is self-adjoint. The operator $xL_h^{(3)}$ is self-adjoint if and only if its coefficients are such that $\alpha_{j+1} = (j/(j+1))\gamma_j$, $j = 1, \cdots, N-1$. The coefficients of $L_h^{(3)}$ are therefore defined by the recursion relations

$$\beta_1 = \frac{4}{3h^2}(\nu+1), \qquad \gamma_1 = -\left(\frac{1}{2}\right)^\nu \frac{4}{3h^2}(\nu+1),$$

$$\alpha_j = \frac{j-1}{j}\gamma_{j-1}, \qquad \beta_j = -\left(\frac{j-1}{j}\right)^\nu \frac{4j}{2j+1}\alpha_j + \frac{4(\nu+1)}{(2j+1)h^2}, \tag{16}$$

$$\gamma_j = \left(\frac{j-1}{j+1}\right)^\nu \frac{(2j-1)(j-1)}{(2j+1)j}\gamma_{j-1} - \frac{4(\nu+1)}{(2j+1)h^2}\left(\frac{j}{j+1}\right)^\nu, \qquad j = 2, \cdots, N.$$

The first order difference equation for γ_j in (16) can be used to generate the sequence of values $\gamma_j, j = 1, \cdots, N$, and this result used to obtain

$$\begin{aligned} \beta_j &= \frac{4(\nu+1)}{j^2h^2}\left[\frac{j^{-2\nu}}{1-j^2/4}\sum_{k=1}^{j-1} k^{2\nu+1} + \frac{1}{2}\frac{1}{1+j/2}\right] \\ &= \frac{4(\nu+1)}{j^2h^2}\left[\left\{j^{-2\nu}\sum_{k=1}^{j-1} k^{2\nu+1} + \frac{1}{2}j\right\} + K_j\right], \qquad j = 1, \cdots, N, \end{aligned} \tag{17}$$

where $|K_j|$ is less than some positive constant K for all values of j.

We now show that

$$\beta_j = \frac{2}{h^2} + \frac{1}{x_j^2}O(1). \tag{18}$$

The quantity in braces in (17) is equal to the estimate of the integral of $x^{2\nu+1}/j^{2\nu}$ from $x = 0$ to $x = j$ by the trapezoid rule applied at each of the $j+1$ equal subdivisions of $0 \leqq x \leqq j$. Since the graph of this positive integrand is concave upward, the estimate is an upper bound on the value of the integral, which is $j^2/(2\nu+2)$. The value of the sum in the braces in (17) is equal to the estimate of the integral of $x^{2\nu+1}/j^{2\nu}$ from $x = \frac{1}{2}$ to $x = j - \frac{1}{2}$ by the midpoint rule applied at each of the j equal subdivisions of $[\frac{1}{2}, j - \frac{1}{2}]$. Since this positive integrand is concave upward, this estimate is a lower bound on the value of the integral. Hence, an upper bound on the quantity in the braces in (17) is $j^2/(2\nu+2) + c_j$, where $|c_j|$ is less than some positive constant c for all j. Therefore, the two bounds on the value in the braces yield (18).

Suppose $y \in \mathscr{F}_\nu$. Then $v(x) = x^{-\nu}y(x)$ is in $C^4[0, 1]$. If we use Taylor's theorem with remainder to expand $v(x_{j+1})$ and $v(x_{j-1})$ about $x = x_j$, and use the fact that $\{\alpha_j\}$ and $\{\gamma_j\}$ are determined so that $L_h^{(3)}(ax^{\nu+2} + bx^\nu) = L(ax^{\nu+2} + bx^\nu)$ on the mesh, we have

$$\begin{aligned} [L_h^{(3)} - L]y(x_j) &= \left(1 - \frac{h^2}{2}\beta_j\right)\left[y''(x_j) - \frac{2\nu+1}{x_j}y'(x_j) + \frac{\nu(\nu+2)}{x_j^2}y(x_j)\right] \\ &\quad - \frac{x_j^{\nu-2}h^3}{6}\left\{\left[-\frac{jh^2}{2}\beta_j + \frac{2(\nu+1)}{h}x_j\right]v'''(x_j) - \frac{h^3j^2}{4}\beta_j v^{\text{iv}}(\xi)\right. \\ &\quad \left. - \frac{h}{4}\left[-\frac{jh^2}{4}\beta_j + \frac{\nu+1}{h}x_j\right][v^{\text{iv}}(\eta) - v^{\text{iv}}(\zeta)]\right\}, \quad j = 1, \cdots, N, \end{aligned} \tag{19}$$

where $x_{j-1} < \zeta < \xi < \eta < x_{j+1}$. But from (18), the fact that

$$y''(x_j) - \frac{2\nu + 1}{x_j} y'(x_j) + \frac{\nu(\nu + 2)}{x_j^2} y(x_j) = O(1)x_j^{2+\nu},$$

and $v'''(x) = xO(1)$, we have from (19), after some manipulation, that

$$\|[L_h^{(3)} - L]y\|_\infty = O(h^2), \qquad y \in \mathscr{F}_\nu. \tag{20}$$

3. Error bounds. We define the vector norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ in the usual way, that is,

$$\|z\|_2 = \left(\sum_{k=1}^N z_k^2\right)^{1/2}, \qquad \|z\|_\infty = \max_k |z_k|;$$

and a matrix norm subordinate to the vector norm $\|\cdot\|_2$, for any real symmetric matrix A, is

$$\|A\|_2 = |\lambda_{\max}|,$$

where $\lambda_{\max}$ is an eigenvalue of A with largest magnitude.

Since the difference eigenvalue problem (3) has homogeneous boundary conditions, the coefficients α_1 and γ_N in (3) can be set equal to zero. Then the eigenvalue problem (2) is equivalent to the eigenvalue problem for the $N \times N$ tridiagonal matrix A whose elements are given by

$$\begin{aligned} a_{j,j+1} &= \gamma_j, \quad a_{j+1,j} = \alpha_{j+1}, \qquad && j = 1, \cdots, N-1, \\ a_{j,j} &= \beta_j, && j = 1, \cdots, N. \end{aligned}$$

Theorem 1 is an extension of Keller's [4, Theorem 5.3.2] result which holds only for A symmetric.

LEMMA 1. *For any $N \times N$ real matrix A, if there exists a positive definite matrix D such that DAD^{-1} is symmetric, then the eigenvalues $\{\Lambda_j\}_{j=1}^N$ of A are real, and for any real number λ and any nontrivial N-vector y,*

$$\min_k |\Lambda_k - \lambda| \leqq \frac{\|(DAD^{-1} - \lambda I)Dy\|_2}{\|Dy\|_2}. \tag{21}$$

Proof. Since, by hypothesis, $B = DAD^{-1}$ is symmetric, the eigenvalues of A are all real. For any nontrivial y, we set $\tau = (A - \lambda I)y$. Then

$$(DAD^{-1} - \lambda I)Dy = (B - \lambda I)Dy = D\tau.$$

If λ is an eigenvalue of A, $\min_k|\Lambda_k - \lambda| = 0$, and hence (21) holds for such a λ.

If λ is not an eigenvalue of A, then $(B - \lambda I)$ is invertible and $Dy = (B - \lambda I)^{-1}D\tau$. Since, by hypothesis, $(B - \lambda I)^{-1}$ is symmetric, we have

$$\begin{aligned} \|Dy\|_2 &\leqq \|(B - \lambda I)^{-1}\|_2 \|D\tau\|_2 \\ &\leqq \max_k \left(\frac{1}{|\Lambda_k - \lambda|}\right) \|D\tau\|_2 \leqq \frac{1}{\min_k |\Lambda_k - \lambda|} \|D\tau\|_2. \end{aligned}$$

Furthermore, since $\|Dy\|_2$ and $\min_k|\Lambda_k - \lambda|$ are nonzero by hypothesis, we obtain

$$\min_k |\Lambda_k - \lambda| \leqq \|D\tau\|_2 / \|Dy\|_2.$$

By the definition of τ, this is the same as (21). This completes the proof of Lemma 1.

LEMMA 2. *For any $N \times N$ real tridiagonal matrix A, for which the products $a_{j,j-1}a_{j-1,j}$, $j = 2, \cdots, N$, are positive, there exists a positive definite matrix D such that DAD^{-1} is symmetric.*

Proof. If D is an $N \times N$ positive definite diagonal matrix, with kth diagonal element denoted by d_k, defined by

$$d_j = \left(\frac{a_{j-1,j}}{a_{j,j-1}}\right)^{1/2} d_{j-1}, \qquad j = 2, \cdots, N, \tag{22}$$

with d_1 an arbitrary positive number, then computation shows that DAD^{-1} is symmetric. This completes the proof of Lemma 2.

We denote the ratio $a_{j-1,j}/a_{j,j-1}$ used in the proof of Lemma 2 by $R_j^{(1)}$, $R_j^{(2)}$, $R_j^{(3)}$ for the matrices whose eigenvalue problems are associated with those of $L_h^{(1)}$, $L_h^{(2)}$ and $L_h^{(3)}$ respectively. From (5), (10), and (16), we have that

$$R_j^{(1)} = \frac{j}{j-1}, \qquad R_j^{(2)} = \left(\frac{j}{j-1}\right)^{1-2\nu} \frac{j+\nu-\frac{1}{2}}{j-\nu-\frac{1}{2}}, \qquad R_j^{(3)} = \frac{j}{j-1}.$$

We obtain an error bound for a set of operators which include these three operators by the following.

LEMMA 3. *For any L_h as in* (3) *which is such that there exist a real number l, a positive integer s and sets of constants $\{a_k\}$, $\{b_k\}$ with $a_k, b_k > -2$ for which*

$$\frac{\gamma_{j-1}}{\alpha_j} = \left(\frac{j}{j-1}\right)^l \prod_{k=1}^{s} \frac{j+a_k}{j+b_k} > 0, \qquad j = 2, \cdots, N,$$

there exist positive constants c_1 and c_2 such that

$$c_1 j^m \leqq d_j^2 \leqq c_2 j^m, \qquad j = 1, \cdots, N,$$

where $D = \operatorname{diag}(d_1, \cdots, d_N)$ is as in (22), *A is the matrix associated with L_h, and $m = l + \sum_{k=1}^{s}(a_k - b_k)$.*

Proof. From (22), we have

$$d_j^2 = d_2^2 \prod_{k=3}^{j} \left(\frac{k}{k-1}\right)^l \prod_{m=1}^{s} \frac{k+a_m}{k+b_m} = d_2^2 j^l 2^{-l} \prod_{k=3}^{j} \prod_{m=1}^{s} \frac{k+a_m}{k+b_m}, \qquad j = 2, \cdots, N.$$

The value of d_1 is arbitrary and we choose it to be $(\alpha_2/\gamma_1)^{1/2}2^{l/2}$ so that $d_2^2 = 2^l$, and hence,

$$d_j^2 = j^l \prod_{k=3}^{j} \prod_{m=1}^{s} \frac{k+a_m}{k+b_m}, \qquad j = 2, \cdots, N. \tag{23}$$

In terms of the gamma function [6, p. 237],

$$\prod_{p=1}^{k} (p+a) = \Gamma(k+a)/\Gamma(a).$$

An application of this to (23) gives

$$d_j^2 = j^l \prod_{m=1}^{s} \frac{\Gamma(j+a_m)\Gamma(2+b_m)}{\Gamma(2+a_m)\Gamma(j+b_m)}. \tag{24}$$

For a given operator L_h, the product $\prod_{m=1}^{s} \Gamma(2 + b_m)/\Gamma(2 + a_m)$ is a given constant K, since $a_m, b_m > -2$.

It follows from Stirling's formula [5, pp. 254–255] that

$$\Gamma(j + a)/\Gamma(j + b) = j^{a-b}(1 + \delta_j),$$

where $\{\delta_j\}$ is a sequence of values which tend to zero as $j \to \infty$. Applying this to (24), we have

$$d_j^2 = Kj^{l+\Sigma}(1 + \delta_j), \qquad j = 2, \cdots, N,$$

$$\Sigma = \sum_{m=1}^{s} (a_m - b_m)$$

and where $\delta_j \to 0$ as $j \to \infty$. We choose $c_1 = K \inf_j(1 + \delta_j)$ and $c_2 = K \sup_j(1 + \delta_j)$, and the proof of Lemma 3 is complete.

LEMMA 4. *For $y_j = J_\nu(\lambda^{1/2}x_j)$, $j = 1, \cdots, N$, $0 < \nu < 1$, and any real positive number λ, and $D = \operatorname{diag}(d_1, \cdots, d_N)$ such that there exists a nonnegative real number m and positive numbers c_3 and c_4 such that*

$$c_3 j^m \leqq d_j \leqq c_4 j^m, \qquad j = 1, \cdots, N, \tag{25}$$

there exists a positive constant K such that $\|Dy\|_2 \geqq KN^{m+1/2}$ for $h = (N + 1)^{-1}$ sufficiently small. Furthermore, for any vector w, $\|Dw\|_2 \leqq c_4 N^{m+1/2}\|w\|_\infty$.

Proof. For a given value of h, we have

$$\|Dy\|_2^2 = \sum_{k=1}^{N} d_k^2 y_k^2 \geqq \sum_{k=N/2}^{N} d_k^2 y_k^2 \geqq \left(\min_{N/2 \leqq k \leqq N} d_k^2\right) \sum_{k=N/2}^{N} y_k^2 \geqq c_3^2 \left(\frac{N}{2}\right)^{2m} \sum_{k=N/2}^{N} y_k^2. \tag{26}$$

We note that

$$\lim_{N \to \infty} \sum_{k=N/2}^{N} y_k^2 (N + 1)^{-1} = \int_{1/2}^{1} [J_\nu(x)]^2\, dx \equiv K_0$$

for some $K_0 > 0$. Hence, if h is chosen small enough, $\sum_{k=N/2}^{N} y_k^2 \geqq \frac{1}{2}K_0(N + 1)$. If h is that small, then we have, from (26),

$$\|Dy\|_2 \geqq c_3(\tfrac{1}{2})^{m+1/2} N^{m+1/2} K_0^{1/2}.$$

Also, for any N-vector w,

$$\|Dw\|_2^2 = \sum_{k=1}^{N} d_k^2 w_k^2 \leqq \max_k d_k^2 \sum_{k=1}^{N} w_k^2 \leqq c_4 N^{2m+1} \|w\|_\infty^2.$$

This completes the proof of Lemma 4.

THEOREM 1. *For any L_h as in (3) which has coefficients such that*

$$\frac{\gamma_{j-1}}{\alpha_j} = \left(\frac{j}{j-1}\right)^l \prod_{k=1}^{s} \left(\frac{j + a_k}{j + b_k}\right), \qquad j = 3, \cdots, N,$$

where $a_k, b_k > -2$ and l is some real number, and for any eigenvalue λ of (2) and corresponding eigenfunction $J_\nu(\lambda^{1/2}x)$, if $m = l + \sum_{k=1}^{s}(a_k - b_k) \geqq 0$, then for h sufficiently small,

$$\min_k |\Lambda_k - \lambda| \leqq C\|(A - \lambda I)y\|_\infty, \tag{27}$$

where C is some positive constant, $\{\Lambda_k\}_{k=1}^N$ are the eigenvalues of the tridiagonal matrix A of coefficients of L_h, and y is the vector $(J_\nu(\lambda^{1/2}x_1), \cdots, J_\nu(\lambda^{1/2}x_N))$.

Proof. By Lemma 2, we know that the hypotheses of the theorem are sufficient for the existence of a positive definite diagonal matrix which symmetrizes A. Lemma 1 implies that

$$\min_k |\Lambda_k - \lambda| \leqq \|D(A - \lambda I)y\|_2 / \|Dy\|_2. \tag{28}$$

By Lemma 3, there exist positive constants c_1 and c_2 such that

$$c_1 j^{m+2} \leqq d_j^2 \leqq c_2 j^{m+2}, \qquad j = 2, \cdots, N.$$

But by hypothesis, $m + 2 \geqq 0$, so that Lemma 4 implies

$$\|Dy\|_2 \geqq KN^{m+1/2}, \qquad \|D(A - \lambda I)y\|_2 \leqq c_2 N^{m+1/2} \|(A - \lambda I)y\|_\infty \tag{29}$$

for some constant K independent of h and h sufficiently small.

We combine (28) and (29) to obtain

$$\min_k |\Lambda_k - \lambda| \leqq \frac{c_2}{K} \|(A - \lambda I)y\|_\infty = C\|(A - \lambda I)y\|_\infty.$$

This completes the proof of Theorem 1.

Theorem 1 can be applied to the difference operators $L_h^{(2)}$ and $L_h^{(3)}$ to obtain the following corollary.

COROLLARY 1. *The eigenvalues $\{\Lambda_k\}_{k=1}^N$ of the tridiagonal matrix A of coefficients of $L_h^{(2)}$ (respectively $L_h^{(3)}$) are such that, for any given eigenvalue λ of (2) and for h sufficiently small,*

$$\min_k |\Lambda_k - \lambda| = O(h^2).$$

4. Numerical experiments. In the numerical experiments conducted in relation to this study, the smallest eigenvalue Λ_1 of the three difference operators $L_h^{(1)}$, $L_h^{(2)}$, and $L_h^{(3)}$ were computed with error no greater than 10^{-8}. In Tables 1 through 3, the amount these approximations differed from the smallest eigenvalue λ_1 of (1) is given for each operator for $\nu = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and $N + 1 = 1/h = 4, 8, 16, 32, 64, 128$. For any two successive values of $N + 1$, a value of the experimental order of convergence (EOC) of the error as given by $\log(e_{(N+1)/2}/e_{N+1})/\log 2$, where $e_{N+1} = |\lambda_1 - \Lambda_1|$, is also tabulated. The experimental order of convergence is the power of h by which the error is decreased, computed from the values of the error at a given value of h and half that value.

The hypotheses of Theorem 1 are satisfied when L_h is any of the three operators discussed in § 2. Hence, we expect an error bound for the eigenvalues of these operators and any eigenvalues of (1) to be given by Theorem 1.

Since by (8), $L_h^{(1)}$ and L are not consistent with respect to the class of functions $\mathscr{F}_\nu$ which contains the eigenfunctions of (1), Theorem 1 does not imply convergence of the eigenvalues of $L_h^{(1)}$ to those of L. The results displayed in Table 1 indicate that such convergence does occur in the case of the smallest eigenvalue of (1), with apparent order of convergence 2ν. Such behavior of the eigenvalues of $L_h^{(1)}$ can be verified by a technique of Weinberger [7].

Corollary 1 is illustrated by the values in Tables 2 and 3.

TABLE 1
Errors and experimental orders of convergence for the smallest eigenvalues of $L_h^{(1)}$ and L

$N+1$	$\nu=\frac{1}{4}$		$\nu=\frac{1}{2}$		$\nu=\frac{3}{4}$	
	$\Lambda_1-\lambda_1$	EOC	$\Lambda_1-\lambda_1$	EOC	$\Lambda_1-\lambda_1$	EOC
4	$1.122E-1$		$1.518E-1$		$-3.828E-1$	
		0.31		-0.41		2.45
8	$9.068E-1$		$2.028E-1$		$-6.988E-2$	
		0.50		0.64		2.80
16	$6.432E-1$		$1.294E-1$		$-1.005E-2$	
		0.53		0.86		6.75
32	$4.447E-1$		$7.113E-2$		$9.310E-5$	
		0.53		0.94		-3.12
64	$3.072E-1$		$3.709E-2$		$8.096E-4$	
		0.53		0.97		0.71
128	$2.131E-1$		$1.892E-2$		$4.946E-4$	

TABLE 2
Errors and experimental orders of convergence for the smallest eigenvalues of $L_h^{(2)}$ and L

$N+1$	$\nu=\frac{1}{4}$		$\nu=\frac{1}{2}$		$\nu=\frac{3}{4}$	
	$\Lambda_1-\lambda_1$	EOC	$\Lambda_1-\lambda_1$	EOC	$\Lambda_1-\lambda_1$	EOC
4	$-3.206E-1$		$-4.970E-1$		$-6.915E-1$	
		1.93		1.98		2.00
8	$-8.427E-2$		$-1.262E-1$		$-1.730E-1$	
		1.98		1.99		2.00
16	$-2.130E-2$		$-3.167E-2$		$-4.334E-2$	
		2.00		2.00		2.00
32	$-5.336E-3$		$-7.925E-3$		$-1.084E-2$	
		2.00		2.00		2.00
64	$-1.335E-3$		$-1.982E-3$		$-2.711E-3$	
		2.00		2.00		2.00
128	$-3.337E-4$		$-4.954E-4$		$-6.779E-4$	

TABLE 3
Errors and experimental orders of convergence for the smallest eigenvalues of $L_h^{(3)}$ and L

$N+1$	$\nu=\frac{1}{4}$		$\nu=\frac{1}{2}$		$\nu=\frac{3}{4}$	
	$\Lambda_1-\lambda_1$	EOC	$\Lambda_1-\lambda_1$	EOC	$\Lambda_1-\lambda_1$	EOC
4	$-2.849E-1$		$-4.970E-1$		$-7.610E-1$	
		1.95		1.98		1.98
8	$-7.356E-2$		$-1.262E-1$		$-1.928E-1$	
		1.97		1.99		2.00
16	$-1.873E-2$		$-3.167E-2$		$-4.833E-2$	
		1.98		2.00		2.00
32	$-4.739E-3$		$-7.925E-3$		$-1.209E-2$	
		1.99		2.00		2.00
64	$-1.194E-3$		$-1.982E-3$		$-3.022E-3$	
		1.99		2.00		2.00
128	$-3.002E-4$		$-4.954E-4$		$-7.555E-4$	

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