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Bessel Difference Systems of Fractional Order*

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1. INTRODUCTION

In this paper we consider the finite difference system

$$4(\nu + 1)(u_1 - u_2/2^\nu)/3h^2 = \Lambda u_1,$$

$$\begin{aligned} -\left(\frac{j}{j-1}\right)^\nu \frac{1-(\nu+\frac{1}{2})/j}{h^2} u_{j-1} + \frac{2u_j}{h^2} - \left(\frac{j}{j+1}\right)^\nu \frac{1+(\nu+\frac{1}{2})/j}{h^2} u_{j+1} \\ = \Lambda u_j, \qquad j = 2,..., N, \end{aligned} \tag{1}$$

$$u_0 = 0, \qquad u_{N+1} = 0,$$

where N is a fixed positive integer, $h = 1/(N+1)$ and $0 < \nu < 1$. This system is a finite difference analog to the Bessel differential system of order ν:

$$\begin{aligned} -y'' - y'/x + \nu^2 y/x^2 = \lambda y, \qquad x \in (0, 1); \\ y(0) = 0, \qquad y(1) = 0. \end{aligned} \tag{2}$$

The eigenvalues of (1) have been shown to converge to the eigenvalues of (2) like h^2 [3, 4].

For $\nu = 0$, (1) is similar to the system treated by Gergen *et al.* [5, 6]. Some of their results are shown to hold for system (1) when $0 < \nu < 1$. Representations are obtained for the exact eigenvalues and eigenfunctions of systems of the form (1), using a technique similar to one employed by Boyer [2] to treat the case $\nu = 0$.

It is convenient for our purposes to consider the matrix eigenvalue problem equivalent to (1). The $N \times N$ tridiagonal matrix $A_{N,\nu}$ which has eigenvalues and eigenvectors identical to the eigenvalues and eigenfunctions of (1) has nonzero elements given by the coefficients of the scheme (1).

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2. Properties of the Eigenvalues

We first consider some properties of the eigenvalues of (1) and find an upper bound for these eigenvalues.

Theorem 1. *The system* (1) *has* N *eigenvalues which are all real, positive, and bounded above by* $4(N+1)^2$.

Proof. We first show that the eigenvalues of $A_{N,\nu} = (a_{i,j})$ are real and that $A_{N,\nu}$ has a complete set of eigenvectors by exhibiting a nonsingular diagonal matrix D such that $DA_{N,\nu}D^{-1}$ is symmetric. With d_i denoting the diagonal element in the i-th row of D, choose $d_1 = 1$ and

$$d_{k+1} = (a_{k,k+1}/a_{k+1,k})^{1/2}\, d_k\,, \qquad k = 1,\ldots, N-1.$$

Then, since

$$\frac{a_{k,k+1}}{a_{k+1,k}} = \left(\frac{k}{k+1}\right)^{2\nu-1} \frac{k+\nu+\frac{1}{2}}{k-\nu+\frac{1}{2}}, \qquad k = 2,\ldots, N-1,$$

each diagonal element of D is well defined and is positive for all $\nu < \frac{3}{2}$. By direct calculation, $DA_{N,\nu}D^{-1}$ is symmetric and tridiagonal.

Next we show that the eigenvalues of A are positive. We introduce the matrix $C_{N,\nu} = D_{N,\nu}^{-1}A_{N,\nu}D_{N,\nu}$ where $D_{N,\nu}$ is the $N \times N$ diagonal matrix defined by

$$D_{N,\nu} = \mathrm{diag}(1, 2^\nu,\ldots, N^\nu).$$

The nonzero elements of $C_{N,\nu}$ are

$$\begin{aligned} c_{1,1} &= 4(\nu+1)/3h^2, \qquad c_{1,2} = -\,4(\nu+1)/3h^2, \\ c_{k,k+1} &= -\,[1+(\nu+\tfrac{1}{2})/k]/h^2, \qquad k = 2,\ldots, N-1, \\ c_{k,k-1} &= -\,[1-(\nu+\tfrac{1}{2})/k]/h^2, \qquad c_{k,k} = 2/h^2, \qquad k = 2,\ldots, N. \end{aligned} \tag{3}$$

We recall that $h = 1/(N+1)$. The matrix $C_{N,\nu}$ is irreducibly diagonally dominant, so that it follows that $C_{N,\nu}$ is nonsingular and all of its eigenvalues have positive real part [8, Theorem 1.8]. Hence, since it has been shown above that the eigenvalues are real, they are each positive.

We now obtain a bound on the eigenvalues of (1) by obtaining a bound on the eigenvalues of $C_{N,\nu}$. If $\nu \leqslant \frac{1}{2}$, then each row sum of $C_{N,\nu}$ is less than or equal to $4/h^2 = 4(N+1)^2$, proving Theorem 1 for $0 < \nu \leqslant \frac{1}{2}$.

In the case $\frac{1}{2} < \nu < 1$, the Sturm sequence, $\{f_j\}$, for the tridiagonal matrix $C_{N,\nu}$ is defined by

$$\begin{aligned} f_0(x) &= 1, \\ f_1(x) &= (x - c_{1,1}) f_0(x), \\ f_{k+1}(x) &= (x - c_{k+1,k+1}) f_k(x) - c_{k+1,k} c_{k,k+1} f_{k-1}(x), \qquad k = 1,..., N-1. \end{aligned} \tag{4}$$

The number of sign changes in the sequence $\{f_j(x)\}$ is equal to the number of eigenvalues of $C_{N,\nu}$ that exceed x [1, p. 203]. By induction we prove that elements of the Sturm sequence have the same sign when $\frac{1}{2} < \nu < 1$ for $x = 4(N+1)^2$, in which case the sequence (4) becomes

$$\begin{aligned} f_0 = 1, \quad f_1 &= 4(2-\nu) f_0/3h^2, \\ f_{k+1} &= 2f_k/h^2 - \left(1 + \frac{\frac{1}{4} - \nu^2}{k^2 + k}\right) f_{k-1}/h^4, \qquad k = 1,..., N-1. \end{aligned} \tag{5}$$

Observe that

$$f_1 = 4(2-\nu)/3h^2 > 1/h^2 = f_0/h^2.$$

Assume, for a given value of k, that $f_k > f_{k-1}/h^2$. Then a computation of f_{k+1}, using (5), gives

$$f_{k+1} > 2f_k/h^2 - \left(1 + \frac{\frac{1}{4} - \nu^2}{k^2 + k}\right) f_k/h^2 > f_k/h^2,$$

where the first inequality follows from the induction hypothesis and the second from the fact that $\nu > \frac{1}{2}$. Hence, we have shown that $f_{k+1} > f_k/h^2$, $k = 0,..., N-1$, and therefore, all eigenvalues of $C_{N,\nu}$ are smaller than $4(N+1)^2$. Since the eigenvalues of $C_{N,\nu}$ are identical to those of (1), the proof of Theorem 1 is complete.

We note that in the proof given above, ν could be any value in the interval $0 < \nu < \frac{3}{2}$; however, we are only interested in the results for $0 < \nu < 1$.

3. Exact Representation of the Solutions

We denote by $P_r^s(x)$ and $Q_r^s(x)$ the associated Legendre functions of degree r and order s of the first and second kinds, respectively. This pair of functions is linearly independent, and linear combinations of them yield the complete solution to the differential equation

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + [r(r+1) - s^2/(1 - x^2)]\, y = 0, \qquad x \in (-1, 1).$$

Many properties of these functions are given by Robin [7]. In particular, two recurrence relations which are useful in obtaining representations for the eigenfunctions of (1) are given in Lemma 1.

LEMMA 1 [7, pp. 163–165]. *Let $Y_r^s(x)$ be any linear combination of the two functions $P_r^s(x)$ and $Q_r^s(x)$. Then*

$$(s - r - 1)\, Y_{r+1}^s(x) + (2r + 1)\, xY_r^s(x) - (s + r)\, Y_{r-1}^s(x) = 0, \qquad (6)$$

$$(x^2 - 1)\frac{dY_r^s(x)}{dx} - (r + 1)\, xY_r^s(x) + (s - r - 1)\, Y_{r+1}^s(x) = 0, \qquad (7)$$

for $-1 < x < 1$ and any real values of r and s.

It is well known that $P_r^s(x)$ and $Q_r^s(x)$ are independent with respect to the variable x for r and s fixed, but we need to establish that they are independent with respect to r for x and s fixed. This is done in the proof of Theorem 2.

THEOREM 2. *The general solution of* (6) *is*

$$Y_r^s(x) = C_1 P_r^s(x) + C_2 Q_r^s(x),$$

where C_1 and C_2 are arbitrary constants and for any r_0, $r = r_0$, $r_0 + 1, \ldots$.

Proof. Since (6) is a second-order linear homogeneous difference equation with independent variable r, we need to show only that $P_r^s(x)$ and $Q_r^s(x)$ are linearly independent as functions of r. We fix s and $x = x_0$. If $P_r^s(x_0)$ and $Q_r^s(x_0)$ were not linearly independent for $r = r_0$, $r_0 + 1, \ldots$, then

$$P_{r_0+k}^s(x_0) = aQ_{r_0+k}^s(x_0) \qquad \text{for } k = 0, 1, \ldots,$$

and some constant a. Then by (7), it would follow that $dP_{r_0}^s/dx = adQ_{r_0}^s/dx$ at the point $x = x_0$. But, by definition, $P_{r_0}^s$ and $Q_{r_0}^s$ are solutions of the same second-order linear homogeneous differential equation on $(-1, 1)$ so that $P_{r_0}^s(x) = aQ_{r_0}^s(x)$ for all x in $(-1, 1)$. But this contradicts the fact that they are linearly independent with respect to x. This completes the proof of Theorem 2.

Set $s = -\nu$ and $r = j - \frac{1}{2}$ in (6) and, for any w in $(0, \pi)$, and any $Y_{j-\frac{1}{2}}^{-\nu}$, we define the function S_j by

$$S_j(\omega) = j^\nu Y_{j-\frac{1}{2}}^{-\nu}(\cos \omega).$$

We now show that with proper choices of $Y_{j-\frac{1}{2}}^{-\nu}$ and ω, we obtain all solutions of (1). With this substitution, (6) becomes

$$\begin{aligned} &-(j-1)^{-\nu}[1-(\nu+\tfrac{1}{2})/j]\, S_{j-1}(\omega) + 2j^{-\nu}\cos\omega S_j(\omega) \\ &-(j+1)^{-\nu}[1+(\nu+\tfrac{1}{2})/j]\, S_{j+1}(\omega) = 0. \end{aligned} \tag{8}$$

A rearrangement of (8) and restricting j to integral values yield that $S_j(\omega)$ is the general solution of

$$-\left(\frac{j}{j-1}\right)^{\nu}\frac{1-(\nu+\frac{1}{2})/j}{h^2}\,u_{j-1} + \frac{2}{h^2}\,u_j - \left(\frac{j}{j+1}\right)^{\nu}\frac{1+(\nu+\frac{1}{2})/j}{h^2}\,u_{j+1}$$

$$= \left(\frac{4}{h^2}\sin^2\frac{\omega}{2}\right) u_j\,.$$

This last system is identical to (1) for $j = 2,\ldots, N$ if

$$\Lambda = (4\sin^2\tfrac{1}{2}\,\omega)/h^2.$$

The function $Y_{j-\frac{1}{2}}^{-\nu}$ in the definition of S_j contains two arbitrary constants, one of which can be determined such that $S_j(\omega)$ satisfies (1) for $j = 1$. Then, since $S_0(\omega)$ is obviously zero, the only remaining property needed for $S_j(\omega)$ to be a solution to (1) is that $S_{N+1}(\omega) = 0$. For any Λ in $(0, 4(N+1)^2)$, there exists an ω in $(0, \pi)$ such that

$$\Lambda = 4(N+1)^2\sin^2\tfrac{1}{2}\,\omega.$$

By Theorem 1, all the eigenvalues Λ_k of the problem (1) lie in $(0, 4(N+1)^2)$, so any eigenfunction of (1) can be represented by $S_j(\omega_k)$, $j = 0,\ldots, N+1$, where S_j is defined by

$$S_j(\omega) = j^{\nu}[C_1 P_{j+\frac{1}{2}}^{-\nu}(\cos\omega) + C_2 Q_{j+\frac{1}{2}}^{-\nu}(\cos\omega)].$$

C_1 and C_2 are related constants, not both zero, one of which is arbitrary, while the other is determined in such a way that $S_j(\omega_k)$ satisfies (1) for $j = 1$. The value of ω_k is related to the k-th eigenvalue of (1) by

$$\omega_k = 2\sin^{-1}[\Lambda_k^{\frac{1}{2}}/2(N+1)].$$

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