APPROXIMATION OF BESSEL'S DIFFERENTIAL OPERATOR OF FRACTIONAL ORDER BY FINITE-DIFFERENCE OPERATORS

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ABSTRACT

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Difference equation problems are studied whose solutions are estimates of the solutions of the two-point boundary-value problem $\mathcal{L}y = f(x)$, $y(0) = 0$, $y(1) = A$ and the eigen-value problem $\mathcal{L}y = \lambda x^2 y$, $y(0) = 0$, $y(1) = 0$, where $\mathcal{L}$ is the Bessel operator of fractional order, $\mathcal{L} = -x^2 d^2/dx^2 - x d/dx + \nu^2$, $0 < \nu < 1$. Three-point difference operators, L_h, are constructed so that $L_h \varphi^{(k)}(x_j) = \mathcal{L}\varphi^{(k)}(x_j)$, $k = 1,2,3$, $j = 1, \ldots, N$, where $\{\varphi^{(k)}\}_{k=1}^{3}$ is a given set of functions and $x_j = j/(N + 1)$.

For differential equation boundary-value problems whose solutions behave like $\alpha x^{\nu} + \beta x^{\nu+2}$ as $x \to 0$ (that is, like the Bessel function J_ν), it is shown that the operator constructed by choosing $\varphi^{(k)} = x^{\nu+k-1}$ and $\mathcal{L}$ have truncation error which is bounded, and the solution of the resulting difference equation problem and the solution of the differential equation problem have discretization error which is $O(h^2)$ as $h \to 0$.

An upper bound on the distance between an eigenvalue, λ, of $\frac{1}{x^2}\mathcal{L}$ and the set $\{\Lambda_j\}_1^N$ of eigenvalues of any three-point difference operator L_h whose coefficients satisfy prescribed conditions, is obtained. The bound is a product of a constant and the max-norm of the truncation error of $\frac{1}{x^2}\mathcal{L}$ and L_h with respect to the eigenfunctions of $\mathcal{L}$ corresponding to λ. Several difference operators are shown to have coefficients which satisfy the prescribed conditions, and for these, $\min_j |\lambda - \Lambda_j|$ is shown to be $O(h^2)$.

Exact representations of the eigenfunctions and eigenvalues of one of the difference operators are obtained in terms of associated Legendre functions.

CHAPTER I

INTRODUCTION

In this work we approximate solutions of problems involving a Bessel differential operator $\mathcal{L}$, which has a single regular singular point at x=0. In particular, we treat the two-point boundary-value problem

$$(1-1a) \qquad \mathcal{L}y(x) = f(x) \qquad , x \in (0,1)$$

$$(1-1b) \qquad y(0) = A, \; y(1) = B,$$

and the eigenvalue problem

$$(1-2a) \qquad \mathcal{L}y(x) = \lambda \rho(x)\, y(x) \qquad , x \in (0,1),$$

$$(1\text{-}2b) \qquad y(0) = 0, \qquad y(1) = 0.$$

We consider only problems for which there is one and only one solution y such that

$$(1\text{-}3) \qquad x^{-m}\, y \in C^4[0,1],$$

where m is some positive number. With this amount of smoothness of the solution, we obtain $O(h^2)$ convergence to y of solutions of certain difference equation problems as the mesh length h tends to zero. If less smoothness were assumed, techniques of analysis would yield a corresponding lower rate of convergence. We treat, in particular, cases in which $0 < m < 1$ so that derivatives of $y(x)$ are unbounded as $x \to 0$.

The example of such an operator which we treat in detail is the Bessel operator of order ν defined by

$$(1\text{-}4) \qquad \mathcal{L}y(x) = -x^2y''(x) - xy'(x) + \nu^2y(x)$$

for $0 < \nu < 1$. In Chapter III we approximate the solution to problem (1-1) with $\mathcal{L}$ defined by (1-4), A=0 in (1-1b), and

$$(1\text{-}5) \qquad x^{-\nu-1}f \in C^4[0,1].$$

In Chapter IV, we approximate the eigenvalue problem (1-2) for $\mathcal{L}$ given by (1-4). In both cases, the solution, y, is of the form (1-3) with $m=\nu$.

We consider approximations to the solution, y, of a problem such as (1-1) obtained by finite-difference approximation on an equally spaced mesh

$$x_j = jh \quad , j = 0,1,\ldots,N+1; \quad h = 1/(N+1) .$$

Second-order difference operators, L, are studied which (in some sense) approximate $\mathcal{L}$ at the interior mesh points. The solution of

$$\text{(1-6a)} \qquad Lu_j = f_j \qquad , j = 1,\ldots,N,$$

$$\text{(1-6b)} \qquad u_o = A, \quad u_{N+1} = B,$$

where $f_j \equiv f(x_j)$, is taken as an estimate of y at the mesh points x_j.

A common method for obtaining L is to replace each derivative in the second-order differential operator $\mathcal{L}$ by a central divided difference. Then if $y \in C^4[0,1]$, $\mathcal{L}y - Ly = O(h^2)$ at each interior mesh point. Keller ([9], p.73) gave an example of such a scheme applied to the problem

$$\text{(1-7a)} \qquad -y''(x) + q(x)\, y'(x) + r(x)\, y(x) = f(x) \quad , a < x < b$$

$$\text{(1-7b)} \qquad y(a) = \alpha \qquad , y(b) = \beta,$$

where q, r, f are continuous on [a,b] and q is positive there. It is shown that for such a difference operator L, the solution u of the corresponding difference equation problem is such that $|u_j - y(x_j)| \leq ch^2$ at each mesh point and for all h sufficiently small, where c is some positive constant independent of h.

An example of eigenvalue approximation was considered by Keller ([9], pp. 131-135) for the problem

(1-8a) $\quad (p(x)\ y'(x))' - q(x)\ y(x) + \lambda r(x)\ y(x) = 0,\ a < x < b,$

(1-8b) $\quad y(a) = 0 \quad , y(b) = 0,$

where $p > 0$, p' and q are continuous, $r > 0$ and $q \geq 0$ on $[a,b]$. He proved that, for the problems (1-8) with eigenfunctions in $C^4[a,b]$, the eigenvalues of the difference operator resulting from replacing all derivatives in (1-8a) by central divided differences, converge to the eigenvalues of (1-8) as h^2, $h \to 0$.

Bulirsch ([4]) extended Keller's result for problem (1-8) to the singular case where q and r each have a pole of first order at either of the endpoints. He proved that the eigenvalues of the difference problem, constructed as above, converge to those of the differential problem (1-8) as h^2, $h \to 0$. Under the above conditions on q and r, the eigenfunctions are in $C^4[a,b]$.

A three-point finite-difference operator L, which approximates a second-order differential operator $\mathcal{L}$ on (0,1) can also be constructed in the following way. Let the functions $\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}$ be a set of functions which is linearly independent on any set of three distinct points in $(0,1)$. Then L is constructed so that

$$(1\text{-}9) \qquad L\varphi^{(k)}(x_j) = \mathcal{L}\varphi^{(k)}(x_j) \qquad ,k = 1,2,3$$

for every mesh point in $(0,1)$, $x_j = jh = j/(N + 1)$.

A common choice of these functions is $\varphi^{(k)}(x) = x^{k-1}$. For $\mathcal{L}$ of the form (1-7a), the L obtained in this way is identical to the one obtained by replacing each derivative by a central divided difference approximation.

Allen ([1]) constructed a difference operator by use of exponential functions for the $\varphi^{(k)}$ and gave some numerical examples. He gave no proofs of convergence.

One expects rapid convergence of the solution of the difference equation to the solution of the differential equation with operators constructed using $\varphi^{(k)}(x) = x^{k-1}$ on an equal spaced mesh, provided that the latter solution can be approximated well over the interval by linear combinations of the $\varphi^{(k)}$. For example, any function with three continuous derivatives can be approximated to order h^3 over an interval of length 2h by a quadratic polynomial. But for problems (1-1) and (1-2) with singular operators whose y are such that

$x^{-\nu}y \in C^4[0,1]$ with $0 < \nu < 1$, local polynomial approximation cannot be guaranteed to be as good as order h^3, as h goes to zero, near the singularity of the derivative at x=0. Consequently, it is doubtful that such difference problems have solutions which converge as fast as h^2, and results of a numerical experiment for such a case are presented in Example 6.2.1 of Chapter VI.

We consider, for the first time, the use of a set of $\varphi^{(k)}$ which does approximate solutions y for which $x^{-\nu}y \in C^4[0,1]$, $0 < \nu < 1$, to order h^3 locally. In Chapter III, we consider the problem which results from replacing the Bessel operator $\mathcal{L}$, as in (1-4), in problem (1-1) by a finite-difference operator constructed as specified by (1-9) with $\varphi^{(k)}(x) = x^{k+\nu-1}$ for $k = 1,2,3$. The function f in (1-1a) is taken to satisfy (1-5). The value A in (1-1b) is chosen to be zero in order to guarantee the boundedness of the solution. It is shown, in the proof of Theorem 3-1, that the solution of this difference problem converges to the solution of the differential problem with order of convergence two for f such that $x^{-\nu-2}f \in C^4[0,1]$.

The Bessel eigenvalue problem (1-2) with $\mathcal{L}$ defined by (1-4) and $p(x) = x^2$, is considered in Chapter IV. The Bessel functions $J_\nu(\lambda^{\frac{1}{2}}x)$ are the eigenfunctions of this problem so that the solution y is of the form

$$(1\text{-}10) \qquad y(x) = \sum_{k=0}^{\infty} c_{2k}\, x^{\nu+2k} \qquad , 0 < x < 1.$$

In Section 4.3 we obtain a bound on the distance between any fixed eigenvalue of the Bessel eigenvalue problem and the set of eigenvalues of <u>any</u> three-point difference operator L_h whose coefficients satisfy given conditions. The bound is given as a product of a constant and the max-norm of the truncation error of L_h and the Bessel operator $\mathcal{L}$. This result shows that if $\mathcal{L}$ and L_h are consistent, then one obtains convergence of their eigenvalues. Furthermore, the rate of convergence is the same as the order of consistency. This result is then applied to one of the difference operators discussed in Chapter III, to obtain a rate of convergence of the eigenvalues of the difference equation eigenvalue problem to the eigenvalues of (1-2). In particular, the problem which results from replacing $\mathcal{L}$ by L_h constructed as in (1-9) with $\varphi^{(k)}(x) = x^{k+\nu-1}$, k = 1,2,3, is shown to have eigenvalues which are convergent to any eigenvalue of (1-2) with order of convergence two.

In Section 4.4, we consider the approximation of the Bessel operator, $\mathcal{L}$, by a finite-difference operator as in (1-9), with $\varphi^{(1)}(x) = x^{\nu}$, $\varphi^{(2)}(x) = x^{\nu+2}$ and $\varphi^{(3)}(x)$ arbitrary. We show that if $\varphi^{(3)}(x) = x^{\mu}$ for any real μ different from ν and $\nu + 2$, then the difference operator and Bessel's operator are consistent with respect to a class of functions which includes all of the eigenfunctions of (1-2), with order of consistency at least two.

For $\varphi^{(3)}$ chosen so that the resulting difference operator is self-adjoint, convergence of the eigenvalues of the difference operator to those of (1-2) with order of convergence at least two is shown in Section 4.5.

Some results are available in the literature on exact representations of eigenfunctions and eigenvalues of Bessel's difference operator of order zero. Gergen, Dressel, and Parrish ([6],[7]) constructed the exact solutions of the eigenvalue problem for the difference operator formed by (1-9) with $\varphi^{(j)} = x^{j-1}$. They showed ([6]) that the eigenvalues of the difference problem converge to those of the differential and that the eigenfunctions also converge ([7]), both with rate of convergence 2.

Another study of approximation of Bessel's operator of non-zero order by finite-differences is given by Pearson ([11]). He has shown, by consideration of the integral representations of the solutions of the difference equation, that the solution of the difference equation formed by choosing $\varphi^{(j)}(x) = x^{j-1}$, converges to the solution of Bessel's equation which is bounded at the origin. Although he does prove convergence, he does not determine the rate of this convergence.

We extend some of the above results on exact representation. In Chapter V we present representations for the eigenvalues and eigenfunctions of the finite difference operator considered in Chapters III and IV which is constructed by choosing $\varphi^{(j)}(x) = x^{\nu+j-1}$, $j = 1,2,3$.

The results of some numerical experiments applying a number of the schemes constructed in Chapter IV are given in Chapter VI.

CHAPTER II

PRELIMINARIES

In this chapter, we describe the class of differential equation problems to which our analysis can be applied, describe the typical difference equation problem and define the terms truncation error, consistency, discretization error and convergence.

2.1 The Differential Operator

We treat the second-order ordinary differential operator $\mathcal{L}$, defined by

$$\mathcal{L}y(x) \equiv p(x)\, y''(x) + q(x)\, y'(x) + r(x)\, y(x) \tag{2-1}$$

where the coefficient functions are of the form $p(x) = \sum_{k=2}^{\infty} p_k x^k$, $q(x) = \sum_{k=1}^{\infty} q_k x^k$, $r(x) = \sum_{k=0}^{\infty} r_k x^k$ on $0 \le x \le 1$. We assume the coefficients are such that the differential operator $\mathcal{L}$ has a regular singular point at the origin, and the origin is the only singular point of $\mathcal{L}$ in $[0,1]$.

We consider the two-point boundary-value problem

$$\mathcal{L}y(x) = f(x) \qquad , x \in (0,1), \tag{2-2a}$$

(2-2b) $\qquad y(0) = A, \quad y(1) = B.$

We now describe the behavior of the solution of (2-2a) for the case which we study in the following chapters.

For the case that the coefficient p_2 is non-zero, direct substitution of

$$y(x) = x^{\nu} \sum_{k=0}^{\infty} c_k x^k$$

into (2-2a) with $f \equiv 0$ yields equations for the coefficients c_k for which this expansion is convergent ([14], p. 197). The value of ν is a root of the indicial equation. Thus, ν is given by

$$\nu = \tfrac{1}{2}\left\{ 1 - \frac{q_1}{p_2} \pm \left[(1 - \frac{q_1}{p_2})^2 - \frac{4r_0}{p_2}\ \right]^{\frac{1}{2}}\right\}.$$

We treat only the case that one of these values is between 0 and 1, and the second is negative. For this case, the series with $0 < \nu < 1$ is zero at $x = 0$ and the series with $\nu < 0$ is singular at $x = 0$. Hence, the use of a zero boundary condition at $x = 0$ excludes singular solutions of the homogeneous equation.

If the right side of the nonhomogeneous equation (2-2a) has the form $f(x) = x^m \sum_{k=0}^{\infty} a_k x^k$ for some nonnegative real number m, and if $a_0 \neq 0$ and $m \neq \nu$, one easily obtains the general solution of (2-2a) which is bounded at x=0 to be

$$y(x) = ax^{\nu} \sum_{k=0}^{\infty} c_k x^k + x^m \sum_{k=0}^{\infty} b_k x^k,$$

where the coefficients b_k can be expressed in terms of the coefficients of p,q,r and f (provided the coefficients a_k tend to zero rapidly enough as $k\uparrow\infty$). With $0 < \nu < 1$, the value of y(0) is zero if $m > 0$ and b_0 if m=0. The boundary condition y(1) = B can be satisfied by proper choice of the value of a.

For $p_2 \neq 0$ and the special case that $m=\nu$, the solution of the nonhomogeneous problem (2-2a) has a logarithmic singularity at x=0 if $a_0 \neq 0$. Hence, for the case $m=\nu$, one obtains a bounded solution only for the case $a_0=0$.

We study problems (2-2) whose solutions behave like $\alpha x^{\nu} + \beta x^{\nu+1} + \gamma x^{\nu+2}$, $0 \le \nu < 1$, as $x \to 0$. Thus, we include the operator $\mathcal{L}$ of (2-1) for appropriate ranges of values p_2, q_1 and r_0 which yield real roots of the indicial equation one of which is negative and the other, ν, satisfies $0 \le \nu < 1$. We include functions f on the right side of (2-2a) of the form $x^m \sum_{k=0} a_k x^k$ for which $m = \nu+1$ or for which $m > \nu+2$ when $0 < \nu \le 1$. For such problems we use the boundary condition y(0) = A = 0 for the cases that $0 < \nu < 1$.

2.2 The Difference Operator

We limit our study to uniform meshes on the interval $[0,1]$ with mesh points x_j,

$$x_j = jh \quad , j = 0,\dots, N+1; \qquad h = 1/(N+1).$$

The norm $||\cdot||$ for any function f defined on the domain D_f, where D_f is either the interval $[0,1]$ or the set of mesh points on $[0,1]$, is defined by

$$||\, f \,|| \equiv \max_{x\in D_f} |f(x)|.$$

The difference operators L_h, which we treat, are of the form

$$\text{(2-4)} \qquad L_h u_j = a_j u_{j-1} + b_j u_j + c_j u_{j+1} \qquad , j = 1,\dots,N,$$

where u_j, $j = 0,1,\dots,N+1$, is any set of real numbers, and have specific choices of the coefficients a, b and c. Each of our choices of coefficients a,b,c in subsequent chapters depends on the mesh length h.

For a given L_h as in (2-4), we can take the solution (if it exists) of

$$\text{(2-5a)} \qquad L_h u_j = f(x_j) \ , \qquad j = 1,\dots,N,$$

$$(2\text{-}5b) \qquad u_0 = A \; , \; u_{N+1} = B$$

as an approximation at mesh points $x = x_j$ of the solution $y(x)$ of (2-2). The solution of (2-5) exists and is unique if the tridiagonal $N \times N$ matrix, A, with elements

$$a_{11} = b_1, \quad a_{kk} = b_k, \quad a_{k,k-1} = a_k, \; a_{k-1,k} = c_k,$$

$k = 2,\ldots,N$, is non-singular.

2.3 Relations Between Differential and Difference Operators

Definition 2-1: For a given set $\mathfrak{F}$ of functions in the domain of $\mathfrak{L}$, the truncation error, τ, [with respect to $\mathfrak{F}$] of the operators $\mathfrak{L}$ and L_h is defined by

$$(2\text{-}6) \quad \tau_j(h\,;\,u) \equiv L_h u(x_j) - \mathfrak{L}u(x_j), \; j = 1,\ldots,N, \; u\in\mathfrak{F}.$$

Definition 2-2: For a given set $\mathfrak{F}$ of functions in the domain of $\mathfrak{L}$, the operators $\mathfrak{L}$ and L_h are said to be consistent [with respect to $\mathfrak{F}$] if and only if

$$\lim_{h\to 0} ||\,\tau(h;u)\,|| = 0 \qquad , u\in\mathfrak{F}.$$

Consistent schemes are said to have order of consistency at least p [with respect to $\mathfrak{F}$], if and only if,

$$||\,\tau(h;u)\,|| = O(h^p) \qquad \text{as } h\to 0, \; u\in\mathfrak{F}.$$

Definition 2-3: The discretization error, e, of the problems (2-2) and (2-5) is defined by

$$e_j(h) \equiv u_j - y(x_j) \qquad ,j = 0,1,\ldots,N+1,$$

where u is the solution of (2-5) and y is the solution of (2-2).

Definition 2-4: The solutions of problems (2-2) and (2-5) are said to be convergent if and only if

$$\lim_{h \to 0} || e(h) || = 0 \qquad .$$

Convergent solutions are said to have order of convergence at least p if and only if

$$|| e(h) || = O(h^p), \qquad \text{as } h \to 0.$$

CHAPTER III

APPROXIMATION OF SOLUTIONS OF A BOUNDARY-VALUE PROBLEM FOR BESSEL'S OPERATOR

In this chapter we study a particular two-point boundary-value problem with Bessel's operator $\mathcal{L} = -x^2 \frac{d^2}{dx^2} - x \frac{d}{dx} + \nu^2$. In Section 3.2, we show that a common, three-point finite-difference approximation has order of consistency less than two over a class of functions containing the solution of the differential equation problem; in contrast, this same difference approximation used for problems involving non-singular differential operators yields order of consistency two. We derive a new, three-point difference operator, $L_h^{(2)}$, and show in Section 3.3 that $\mathcal{L}$ and $L_h^{(2)}$ have order of consistency at least two over a class of functions containing the solution of the differential problem. The linear difference operator is shown, in Section 3.4, to be monotonic, and this, together with the fact that $\mathcal{L}$ and $L_h^{(2)}$ are consistent with order of consistency two (Section 3.3), is used to show that the discretization error is order h^2.

3.1 The Differential Equation Problem

In this chapter we study approximation by means of finite differences to solutions of problems of the form

(3-1a) $$\mathcal{L}y = f \qquad ,0 < x < 1$$

(3-1b) $$y(0) = 0, \qquad y(1) = A$$

where $\mathcal{L}$ is the Bessel operator defined by

(3-2) $$\mathcal{L}y(x) = -x^2y''(x) - xy'(x) + \nu^2y(x)$$

for some fixed ν in $(0,1)$.

Particular problems which are of interest, and for which one might wish to attempt approximation by finite-differences, are Bessel's equation

(3-3a) $$\mathcal{L}y = x^2y \qquad ,0 < x < 1,$$

(3-3b) $$y(0) = 0, \; y(1) = A$$

and the Bessel eigenvalue problem

(3-4a) $$\mathcal{L}y = \lambda x^2y \qquad ,0 < x < 1,$$

(3-4b) $$y(0) = 0 \; , \; y(1) = 0$$

and we want to include these special cases in our analysis.

The solution of (3-3) ([14], p.358), is a multiple of the function

$$J_\nu(x) = x^\nu \sum_{k=0}^{\infty} c_{2k} x^{2k}, \quad c_{2k} = (-1)^k \frac{1}{k!\,\Gamma(\nu+k+1)\, 2^{\nu+2k}},$$

and the eigenfunctions of (3-4) are constant multiples of the functions $J_\nu(\lambda^{\frac{1}{2}}x)$ for all eigenvalues, λ, of (3-4). The right sides of (3-3a) and (3-4a) are, therefore, the products of an analytic function and $x^{\nu+2}$.

We would like to include in our analysis functions f of the form

$$(3\text{-}5) \qquad f(x) = x^\nu \sum_{k=0}^{\infty} a_k x^k.$$

But if $a_0 \neq 0$, then the solution of (3-1a) has a logarithmic singularity at x=0. Hence, we consider only right sides f such that $x^{-\nu} f \to 0$ as $x \to 0$.

The general solution of the homogeneous equation

$$(3\text{-}6) \qquad \mathcal{L}y(x) = 0 \qquad , 0 < x < 1,$$

is $ax^\nu + bx^{-\nu}$ for any constants a and b. For problem (3-1) with f as in (3-5) with $a_0 = 0$ and the boundary condition $y(0) = 0$, if the solution $y(x)$ is

$$(3\text{-}7) \qquad y(x) = x^\nu \sum_{k=0}^{\infty} c_k x^k \qquad , 0 < x < 1,$$

then direct substitution yields c_0 arbitrary and $c_k = (k^2+2\nu k)a_k$, $k = 2,3,\ldots$. The value of c_0 is determined by the second boundary condition $y(1) = A$. The series (3-7) converges if, and only if, $\sum_{k=1}^{\infty} a_k(k^2 + 2\nu k)$ converges.

Our analysis applies to problems (3-1) in which the right side has the form

$$f(x) = \sum_{k=1}^{3} a_k x^{\nu+k} + g(x) \quad , \qquad 0 \le x \le 0,$$

where $g \in x^{\nu+4} C^4[0,1]$, that is $x^{-\nu-4} g \in C^4[0,1]$. For this f, the solution, y, of the problem has the form

$$y(x) = \sum_{k=0}^{3} c_k x^{\nu+k} + G(x)$$

where $G \in x^{\nu+4} C^4[0,1]$. We denote such a function as an element of the family $\mathfrak{F}$, that is,

$$\text{(3-8)} \quad \mathfrak{F} = \left\{ y \,\middle|\, y(x) = \sum_{k=0}^{3} c_k x^{\nu+k} + G(x), \;\; G \in x^{\nu+4} C^4[0,1], \; 0 \le x \le 1 \right\}$$

We note that $J_\nu \in \mathfrak{F}$ for any positive ν and for J_ν, $c_1 = c_3 = 0$.

3.2 The Difference Equation Problem

A common technique for obtaining a difference approximation to a differential operator $\mathfrak{L}$ is to replace all derivatives by standard $O(h^2)$ divided-difference approximations to them on a uniform mesh (c.f. Chapter I). For $\mathfrak{L}$ defined by (3-2), the difference operator, $L_h^{(1)}$, obtained in this way is

$$\text{(3-9)} \quad L_h^{(1)} u_j = -x^2 \left[\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \right] - x_j \left[\frac{-u_{j-1} + u_{j+1}}{2h} \right] + \nu^2 u_j$$

$$= (-j^2 + \tfrac{1}{2}j)\, u_{j-1} + (2j^2 + \nu^2)\, u_j + (-j^2 - \tfrac{1}{2}j)\, u_{j+1}, \quad j=1,\ldots,N.$$

We investigate the truncation error of $\mathcal{L}$ and $L_h^{(1)}$ for $y \in \mathfrak{F}$, where $\mathfrak{F}$ is given by (3-8). For such a y,

(3-10)

$$L_h^{(1)}\, y(x_j) = \mathcal{L}y(x_j) + h^2 \left\{ -\frac{x_j^2}{24}\, y^{(4)}\,(\xi) - \frac{x_j}{6}\, y^{(3)}(x_j) \right\} + O(h^3),$$

where $x_{j-1} < \xi < x_{j+1}$. Since y(x) behaves like $x^{\nu} + O(x^{\nu+1})$ as x goes to zero, then as h goes to zero and j remains fixed, we note that $y^{(4)}(x_j) = O(h^{\nu-4})$ and $y^{(3)}(x_j) = O(h^{\nu-3})$. Therefore,

$$(3\text{-}11) \qquad L_h^{(1)} y(x_j) = \mathcal{L}y(x_j) + O(h^{\nu}) \qquad , h \to 0,\ j \text{ fixed.}$$

Away from zero, the derivatives of y(x) are bounded, since y(x) is in $C^4(0,1)$. Hence, for any positive number c in (0,1), $L_h^{(1)} y(x_j) = \mathcal{L}y(x_j) + O(h^2)$, $h \to 0,\ x_j > c > 0$. By (3-11), $\mathcal{L}$ and $L_h^{(1)}$ are consistent with respect to $\mathfrak{F}$, with order of consistency at least ν.

To show that the above order of consistency is exactly ν, we use the function $y(x) = x^{\nu}$. At the first mesh point, j = 1, $L_h^{(1)}$ is

$$L_h^{(1)}\, x_1^{\nu} = [-3 \times 2^{\nu-1} + 2 + \nu^2]\, h^{\nu}.$$

The value in the brackets in the above expression is always non-zero. Since x^{ν} is a solution of the homogeneous equation,

$\mathcal{L}x^{\nu} = 0$. Hence, $L_h^{(1)} x^{\nu}$ is equal to the truncation error at j=1 and so there exists at least one function, namely x^{ν}, for which the order of consistency is exactly ν. Since the order is exactly ν for $y(x) = x^{\nu}$, it is also exactly ν for any function whose series includes a multiple of x^{ν} as a term. For example, this is true of $y(x) = J_{\nu}(\lambda^{\frac{1}{2}}x)$ for any positive real λ.

In Chapter IV, we show that for the eigenvalue problem (4-1), the eigenvalues of a difference problem obtained by replacing the differential operator in the eigenvalue problem by an approximating difference operator, converge to the eigenvalues of (4-1) with error bounded by the order of consistency of the difference and differential operator with respect to the eigenfunctions of (4-1), which are all of the form $J_{\nu}(\lambda^{\frac{1}{2}}x)$ for some λ. When the difference operator used is $L_h^{(1)}$, we hence have a bound $O(h^{\nu})$. This behavior of the truncation error apparently explains the relatively slow convergence of the eigenvalues and eigenfunctions observed in numerical experiments (see Chapter VI Example 6.2.1).

The reason that $L_h^{(1)}$ and $\mathcal{L}$ fail to be consistent with order of consistency at least two, which is the order obtained with difference approximation to a non-singular operator, is that the usual difference approximation is constructed in such a way that at mesh points, the values of $\mathcal{L}p(x_j)$, where p(x) is any quadratic polynomial, equals $L_h^{(1)}p(x_j)$, that is,

$$L_h^{(1)}p(x_j) = \mathcal{L}p(x_j) \qquad ,j = 1,\ldots,N.$$

One expects such approximations to yield good estimates of solutions which do behave like polynomials locally, since quadratic polynomial interpolation on equally spaced mesh has local accuracy $O(h^3)$. But solutions of the form (3-7) cannot be approximated locally to $O(h^3)$ by quadratic polynomials near x=0.

In general, a three-point difference operator can be constructed to approximate a differential operator $\mathcal{L}$ for any given set of three functions $\{\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}\}$, linearly independent on the set of mesh points $x_j = jh$, $j = 1,\ldots,N+1$, so that

$$L_h\{\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}\}\,\varphi^{(k)}(x_j) = \mathcal{L}\varphi^{(k)}(x_j) \quad ,j = 1,\ldots,N,\ k = 1,2,3.$$

Given three such functions, one can obtain the coefficients of the difference operator, L_h, given by

$$L_h u_j = \alpha_j u_{j-1} + \beta_j u_j + \gamma_j u_{j+1} \qquad ,j = 1,\ldots,N,$$

by solving the system of equations

(3-12)

$$\alpha_j \varphi^{(k)}(x_{j-1}) + \beta_j \varphi^{(k)}(x_j) + \gamma_j \varphi^{(k)}(x_{j+1}) = \mathcal{L}\varphi^{(k)}(x_j),\ k = 1,2,3.$$

In particular, in order to approximate the solution of (3-1) when the solution is in class $\mathfrak{F}$, we approximate $\mathcal{L}$ by a difference operator constructed as above with $\varphi^{(k)}(x) = x^{\nu+k-1}$, $k = 1,2,3$. Note that when $j = 1$, the system (3-12) reduces to a system of three equations in two unknowns. Hence, at this point only two of the functions x^{ν}, $x^{\nu+1}$, $x^{\nu+2}$ can be used to construct the approximating operator. We choose x^{ν} and $x^{\nu+2}$. This operator, which we investigate in this chapter, we denote by $L_h^{(2)}$. We use the notation

(3-13a) $$L_h^{(2)} u_1 = L_h\{x^{\nu}, x^{\nu+2}\}\, u_1$$

(3-13b) $$L_h^{(2)} u_j = L_h\{x^{\nu}, x^{\nu+1}, x^{\nu+2}\} u_j \qquad , j = 2,\dots,N,$$

for any set of numbers $\{u_j\}$.

Computation of the coefficients by solving (3-12) gives $L_h^{(2)}$ as

(3-14a) $$L_h^{(2)} u_1 = (4\nu + 4)\ [u_1 - (\tfrac{1}{2})^{\nu} u_2]/3$$

(3-14b)

$$L_h^{(2)} u_j = \left(\frac{j}{j-1}\right)^{\nu} (-j^2 + \nu j + \tfrac{1}{2} j)\, u_{j-1} + 2j^2 u_j + \left(\frac{j}{j+1}\right)^{\nu} \cdot$$
$$\cdot\, (-j^2 - \nu j - \tfrac{1}{2} j)\, u_{j+1}, \qquad j = 2,\dots,N.$$

In terms of $x_j = jh$, this can be written

$$L_h^{(2)} u_j = -\frac{x_j^2}{h^2}\left[\left(\frac{x_j}{x_{j-1}}\right)^{\nu} u_{j-1} - 2u_j + \left(\frac{x_j}{x_{j+1}}\right)^{\nu} u_{j+1}\right] -$$

$$\frac{x_j}{h}\left[\left(\frac{x_j}{x_{j-1}}\right)^{\nu} - \left(\frac{x_j}{x_{j+1}}\right)^{\nu}\right] [\nu + \tfrac{1}{2}], \; j = 2,\ldots,N.$$

If we define the operator $\mathcal{K}$ by

$$\text{(3-15)} \quad \mathcal{K}y(x) = -x^2 y''(x) - x(2\nu + 1)\, y'(x) \quad ,0 < x < 1,$$

then $\mathcal{L}x^{\nu}y(x) = x^{\nu}\mathcal{K}y(x)$. If we approximate $\mathcal{K}$ by replacing each derivative by an $O(h^2)$ divided-difference approximation, we obtain K_h given by

(3-16a)

$$K_h u_j = -x_j^2\left(\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}\right) - x_j(2\nu + 1)\left(\frac{-u_{j-1} + u_{j+1}}{2h}\right)$$

$$= (-j + j\nu + \tfrac{1}{2}j)\, u_{j-1} + 2j^2\, u_j + (-j^2 - j\nu - \tfrac{1}{2}j)\, u_{j+1}, \; j = 1,\ldots,N,$$

for any set of real numbers $\{u_j\}$. We note that $L_h^{(2)} j^{\nu} u_j = j^{\nu} K_h u_j$, for $j = 2,\ldots,N$. Hence we might expect $L_h^{(2)}$ to be a good approximation to $\mathcal{L}$ on $\mathcal{S}$ since K_h is a good approximation to $\mathcal{K}$ on $C^4[0,1]$.

As we noted above, the operators K_h and $L_h^{(2)}$ are related by $L_h^{(2)} u_j = j^{\nu} K_h u_j$; j=2,...,N. To obtain an operator for which this correspondence holds even at j=1, we define $K_h^{(2)}$ by

$$\text{(3-16b)} \quad \begin{aligned} K_h^{(2)} u_1 &= x_1^{-\nu} L_h^{(2)} \left[(x^{\nu} u)_1 \right] = (4\nu + 4)\ (u_1 - u_2)/3, \\ K_h^{(2)} u_j &= K_h u_j \qquad\qquad\qquad\qquad , \ j=2,\ldots,N. \end{aligned}$$

For three-point difference equation problems with given values of u at the boundary points j=0 and j=N + 1, one has an equivalent tridiagonal matrix problem. We prove the following lemma which we need in Chapter V.

<u>Lemma 3-1</u>: The matrices A and C, equivalent to the difference operators $\frac{1}{x_j^2} L_h^{(2)}$ and $\frac{1}{x_j^2} K_h^{(2)}$, respectively, are similar. The matrix C is non-singular, and all eigenvalues of C have positive real part.

<u>Proof</u>: If the matrix D is defined by

$$D = \mathrm{diag}(1, 2^{\nu}, \ldots, N^{\nu}),$$

then $C = D^{-1}AD$. The matrix C is irreducibly diagonally dominant, which can be seen from (3-16). Therefore, C is nonsingular and all of its eigenvalues have positive real part ([13], Theorem 1.8, p.23).

3.3 Truncation Error Analysis

Lemma 3-2: For $\mathcal{L}$ and $L_h^{(2)}$ given by (3-2) and (3-14) and $u \in \mathfrak{F}$ so that $u(x) = \sum_{k=0}^{3} c_k x^{\nu+k} + G(x)$, $G \in x^{\nu+4} C^4[0,1]$,

$$(3\text{-}17) \quad (\mathcal{L} - L_h^{(2)})\, u(x_1) = c_1 O(h^{\nu+1}) + (c_3 x_1^{\nu+1} + x_1^{\nu+2})\, O(h^2)$$

$$(\mathcal{L} - L_h^{(2)})\, u(x_j) = (c_3 x_j^{\nu+1} + x_j^{\nu+2})\, O(h^2),\ j = 2,\dots,N.$$

Proof: For $u(x)$ as given, we have, by the construction of $L_h^{(2)}$, that

$$(\mathcal{L} - L_h^{(2)})\, u(x_1) = (\mathcal{L} - L_h^{(2)})\, [\sum_{k=0}^{3} c_k x_j^{\nu+k} + G(x_1)]$$

Since $\mathcal{L} x^{\nu+k} = [\nu^2 - (\nu + k)^2]\, x^{\nu+k}$ and $L_h^{(2)} x_1^{\nu+k} =$

$$= \frac{h^{k+\nu}}{3} \quad (4\nu + 4)\ (1 - 2^k)$$

we have $\mathcal{L}$ and $L_h^{(2)}$ of $G(x_1)$ are both $O(h^{\nu+4})$ since $G \in x^{\nu+4} C^4[0,1]$, and hence

$$(\mathcal{L} - L_h^{(2)})\, u(x_1) = \sum_{k=0}^{3} c_k [\nu^2 - (\nu + k)^2] h^{\nu+k} -$$

$$- \sum_{k=0}^{3} c_k \frac{4}{3}(\nu+1)\ (1-2^k)\ h^{\nu+k} + O(h^{\nu+4}) =$$

$$= c_1\ \frac{1}{3}(-2\nu + 1)\ h^{\nu+1} + c_3\ \frac{1}{3}(10\nu + 1) + O(h^{\nu+4}) =$$

$$= c_1 h^{\nu+1} + (c_3 x_1^{\nu+1} + x_1^{\nu+2})\ O(h^2)$$

Hence, (3-17) holds at $j = 1$.

Since K_h is constructed by replacing the derivatives in $\mathcal{K}$, defined by (3-15), by the usual divided-difference approximations, then

$$(\mathcal{K} - K_h)\ v(x_j) = \frac{2\nu+1}{6}\, x_j h^2 v^{(3)}(x_j) + \frac{x_j^2}{2}\, h^2 v^{(4)}(\xi) + O(h^3),$$

$$j = 1,\ldots,N,$$

for $x_{j-1} < \xi < x_{j+1}$ and for any function $v(x) \in C^4[0,1]$. If $v(x)$ is given by

$$v(x) = x^{-\nu}u(x) = \sum_{k=0}^{3} c_k x^k + x^{-\nu}G(x)\ , \quad 0 < x < 1,$$

we note that $v^{(3)}(x) = c_3 + xg(x)$, for some function g in $C^1[0,1]$ and $v^{(4)} \in C[0,1]$. This follows since $G \in x^{\nu+4}C^4[0,1]$. From this we have

$$(\mathcal{K} - K_h)\ v(x_j) = c_3 x_j O(h^2) + x_j^2\, O(h^2), \quad j = 1,\ldots,N.$$

Since $\mathcal{L}x^\nu v(x) = x^\nu \mathcal{K}v(x)$ and $L_h^{(2)}\ x_j^\nu v(x_j) = x_j^\nu K_h v(x_j)$ for as noted in Section 3.2, we have

$$(\mathcal{L} - L_h^{(2)})\ u(x_j) = x_j^\nu(\mathcal{K} - K_h)\ v(x_j) = (c_3 x_j^{\nu+1} + x_j^{\nu+2})\ O(h_2^2),$$

$$j = 2,\ldots,N.$$

This completes the proof of Lemma 3-2.

3.4 Proof that $L_h^{(2)}$ is a Monotonic Operator

A linear operator, T, is said to be a monotonic operator if Tu>0 implies that u>0 for arbitrary u in the domain of T. A non-singular linear operator which has a positive inverse is monotonic since, if $Tu = v>0$ then $u = T^{-1}v$ and if $T^{-1}>0$, then $T^{-1}v>0$.

Lemma 3-3: If $L_h^{(2)}v_j>0$ for $j = 1,\ldots,N$ and $v_0 = v_{N+1} = 0$, then $v_j>0$ for $j = 1,\ldots,N$.

Proof: Since the matrix A equivalent to the difference operator $L_h^{(2)}$ is similar to a matrix which is irreducibly diagonally dominant (Lemma 3-1), $L_h^{(2)}$ is non-singular. We show that its inverse is positive and therefore that $L_h^{(2)}$ is monotonic. The matrix A^{-1} has columns with elements $a_{kj} = u_j^{(k)}$, where $u_j^{(k)}$, $j = 1,\ldots,N$, is the solution to the problem

$$\text{(3-18a)} \quad L_h^{(2)}u_j^{(k)} = \delta_j^{(k)} \quad , \quad j = 1,\ldots,N,$$

$$\text{(3-18b)} \quad u_0 = 0 \ , \ u_{N+1} = 0,$$

where $\delta_j^{(k)} = 1$ if $j = k$ and $\delta_j^{(k)} = 0$ is $j \neq k$.

Since $\mathcal{L}x^{\nu} = 0$ and $L_h^{(2)}$ was constructed so that $\mathcal{L}x_j^{\nu} = = L_h^{(2)} \quad x_j^{\nu}$, it follows that $u_j = x_j^{\nu}$ is one solution of $L_h^{(2)}u = 0$. Futhermore, x_j^{ν} satisfies the boundary condition

$u_0 = 0$. Therefore, we have that

$$u_j^{(k)} = a x_j^{\nu} \ , \qquad j = 0, \ldots, k,$$

for some constant a.

If a function f(x) is not identically constant on the mesh, then the pair of functions x_j^{ν} and $x_j^{\nu}\, f(x_j)$ are linearly independent on the mesh. If $x_j^{\nu}\, f(x_j)$ satisfies $L_h y = 0$, then by direct substitution and simplification, we obtain the following difference equation which is satisfied by f:

$$-(j - \nu - \tfrac{1}{2})\, f_{j+1} + 2j f_j - (j + \nu + \tfrac{1}{2})\, f_{j+1} =$$

$$= (j - \nu - \tfrac{1}{2})(f_j - f_{j-1}) + (j + \nu + \tfrac{1}{2})(f_j - f_{j+1}) = 0,$$

$$j = 2, \ldots, N$$

Since $0 < (j - \nu - \tfrac{1}{2}) < (j + \nu + \tfrac{1}{2})$ for $j \geq 2$, and $0 < \nu < 1$, then $f_j - f_{j-1} > f_{j+1} - f_j$. Also, $f_j - f_{j-1}$ and $f_{j+1} - f_j$ have the same sign. Therefore f is a strictly monotone function of j. We take $f_1 = 1$ and $f_2 = 2$. Then f is a positive, strictly monotone increasing function. Thus, with this f, the general solution of (3-18) can be written as

$$u_j^{(k)} = \begin{cases} a(jh)^{\nu} & j \leq k \\ (jh)^{\nu}\,[b + c f_j] & j \geq k \end{cases} .$$

To satisfy the boundary condition $u_{N+1}^{(k)} = 0$, we take $b = -cf_{N+1}$. For both definitions of $u_j^{(k)}$ to be consistent at $j = k$, we take $a = c(f_k - f_{N+1})$. The fact that $L_h^{(2)} u_k^{(k)} = 1$ requires that $c = [k^\nu h^\nu (k^2 + \nu k + \frac{1}{2}k)(f_k - f_{k+1})]^{-1}$, which is negative since $f_{N+1} > f_k$. The coefficient a is positive since it is equal to the product, $c(f_k - f_{N+1})$, of two negative numbers. Therefore, $u_j^{(k)} > 0$ for $j = 1,\ldots,k$. When $j > k$, we have

$$u_j^{(k)} = (jh)^\nu \, c(f_j - f_{N+1}) > 0.$$

This completes the proof of Lemma 3-3.

3.5 An Error Bound

We define the two-point boundary-value problem for the difference operator $L_h^{(2)}$, which is analogous to (3-1), by

(3-19a) $\quad L_h^{(2)} u_j = f(x_j)$

(3-19b) $\quad u_0 = 0, \quad u_{N+1} = A$

where $L_h^{(2)}$ is as in (3-14). In the proof of Theorem 3-1, we show that the solution of (3-19) converges to the solution of (3-1) as $O(h^2)$ when the latter solution is in $\mathfrak{F}$ with $c_1 = 0$.

Theorem 3-1: The solution of (3-19) converges to the solution, y, of (3-1) with order of convergence at least 2 when $y \in \mathfrak{F}$ and $c_1 = 0$.

Proof: We first define the mesh function g by

$$g_j = 2(jh)^\nu - (jh)^{\nu+1}, \quad j = 0,1,\ldots,N+1.$$

We note that

$$L_h^{(2)}\ g_1 = \frac{1}{3}(4\nu + 4)\ (jh)^{\nu+1} ,$$

$$L_h^{(2)}\ g_j = \mathcal{L}(2x_j^{\nu} - x_j^{\nu+1}) = (2\nu + 1)\ (jh)^{\nu+1} \quad , j > 1.$$

So g is a positive function on $j = 1,\ldots,N$ for which $L_h^{(2)}\ g_j > 0$.

The discretization error, e, is

$$e_j = y(x_j) - u_j \quad , \qquad j = 0,1,\ldots,N + 1.$$

Then e is the solution of the problem

$$L_h^{(2)}\ e_j = \tau_j(h;y) \quad , \qquad j = 1,\ldots,N$$
$$e_0 = 0, \quad e_{N+1} = 0$$

For convenience, we drop the parameters of τ_j throughout the remainder of the proof since they are always h and y.

By Lemma 3-2, we have, since $c_1 = 0$,

$$\tau_j = x_j^{\nu+1}\ O(h^2) \quad , \qquad j = 1,\ldots,N.$$

We define σ by $\sigma_j \equiv \dfrac{1}{x_j^{\nu+1}}\ \tau_j$ and therefore $\sigma_j = O(h^2)$. We note that e is also the solution of

$$\frac{1}{x_j^{\nu+1}} L_h^{(2)} e_j = \sigma_j \quad , \qquad j = 1,\ldots,N,$$

$$e_0 = 0, \quad e_{N+1} = 0.$$

We define w by $w_j \equiv ||\sigma||\, g_j - e_j$. Since $\frac{1}{x_j^{\nu+1}} L_h^{(2)} g_j > 1$, for $j = 1,\ldots,N$,

$$\frac{1}{x_j^{\nu+1}} L_h^{(2)} w_j > ||\sigma|| - \sigma_j \geq 0 \quad , \qquad j = 1,\ldots,N.$$

Therefore, by Lemma 3-3, we have

$$\text{(3-20)} \qquad w_j = ||\sigma||\, g_j - e_j > 0 \quad , \qquad j = 1,\ldots,N.$$

We define v by $v_j \equiv ||\sigma||\, g_j + e_j$, we once again obtain

$$\frac{1}{x_j^{\nu+1}} L_h^{(2)} v_j > 0 \quad , \qquad j = 1,\ldots,N$$

and, by Lemma 3-2,

$$\text{(3-21)} \qquad v_j = ||\sigma||\, g_j = e_j > 0 \quad , \qquad j = 1,\ldots,N.$$

Combining (3-20) and (3-21), we have

$$|e_j| < ||\sigma|| \, g_j \leq ||\sigma|| \;\; ||g|| \;, \quad j = 1,\ldots,N.$$

But since $||\sigma||$ is order h^2 and $||g||$ is 2, then

$$||e|| = O(h^2)$$

and the proof is completed.

We observe that if $x^{-2-\nu} f \epsilon C^4[0,1]$, then the solution, y, of (3-1) is of the necessary form for Theorem 3-1 to apply.

CHAPTER IV

BESSEL EIGENVALUE PROBLEM

In this chapter we consider approximations to the eigenvalues of the Bessel eigenvalue problem

(4-1a) $\mathcal{L}y(x) = -x^2y''(x) - xy'(x) + \nu^2 y(x) = \lambda x^2 y(x),$
$0 < x < 1,$

(4-1b) $y(0) = 0, \quad y(1) = 0.$

We take as estimates of eigenvalues λ of (4-1), the eigenvalues, Λ, of

(4-2a) $L_h u_j = \Lambda x_j^2 u_j \quad , \qquad j = 1,\dots,N,$

(4-2b) $u_0 = 0 \quad , u_{N+1} = 0,$

where L_h is a difference operator given by

(4-3) $L_h u_j = \alpha_j u_{j-1} + \beta_j u_j + \gamma_j u_{j+1} \quad , \quad j = 1,\dots,N$

for any set of numbers u_j, and some given set of values α, β, γ.

We note that since the eigenvalue problem (4-2) has homogeneous boundary conditions, the coefficients α_1 and γ_N can be set equal to zero. Then the eigenvalue problem (4-2) is equivalent to the eigenvalue problem for the $N \times N$ tridiagonal matrix A whose elements are the coefficients of $\frac{1}{x_j^2} L_h$. That is, if Λ and u are an eigenvalue and corresponding eigenfunction of (4-2), then

$$Au = \Lambda u \quad .$$

4.1 A General Bound from Matrix Theory

The vector norm $||\cdot||_2$ defined by

$$||z||_2 = \left(\sum_{k=1}^{N} z_k^2 \right)^{\frac{1}{2}}$$

is used throughout this chapter. A matrix norm subordinate to this vector norm is, for any real symmetric matrix A,

$$||A||_2 = |\lambda_{max}|,$$

where λ_{max} is an eigenvalue of A with largest magnitude.

We now prove the following.

Lemma 4-1: For any $N \times N$ real matrix A, if there exists a positive definite matrix D such that DAD^{-1} is symmetric, then the eigenvalues $\{\Lambda_j\}_{j=1}^{N}$ of A are real, and for any real number λ and any non-trivial N-vector y,

$$\min_k |\Lambda_k - \lambda| \le \frac{||(DAD^{-1} - \lambda I)\, Dy||_2}{||Dy||_2} .$$

<u>Proof</u>: Since, by hypothesis, $B = DAD^{-1}$ is symmetric, the eigenvalues $\{\Lambda_j\}$ of A are all real. For any non-trivial vector y, we set

$$\tau = (A - \lambda I)\, y.$$

Then

$$(DAD^{-1} - \lambda I)\, Dy = (B - \lambda I)\, Dy = D\tau.$$

If λ is not an eigenvalue of A, then $(B - \lambda I)$ is invertible and

$$Dy = (B - \lambda I)^{-1}\, D\tau .$$

Since, by hypothesis, $(B - \lambda I)^{-1}$ is symmetric, we have

$$||Dy||_2 \leq ||(B - \lambda I)^{-1}||_2 ||D\tau||_2 \leq$$

$$\leq \max_k \left(\frac{1}{|\Lambda_k - \lambda|} \right) ||D\tau||_2 \leq \frac{1}{\min_k |\Lambda_k - \lambda|} ||D\tau||_2 \quad .$$

Furthermore, since $||Dy||_2$ and $\min_k |\Lambda_k - \lambda|$ are non-zero by hypothesis, we obtain

$$(4\text{-}4) \qquad \min_k |\Lambda_k - \lambda| \leq \frac{||D\tau||_2}{||Dy||_2} = \frac{||(DAD^{-1} - \lambda I)\, Dy||_2}{||Dy||_2} \quad .$$

If λ is an eigenvalue of A, the $\min_k |\Lambda_k - \lambda| = 0$, so (4-4) holds for any λ and any non-trivial y. This completes the proof of Lemma 4-1.

Lemma 4-2: For any $N \times N$ real tridiagonal matrix, A, for which the product $a_{k,k-1}a_{k-1,k}$ is positive, $k = 2,\ldots,N$, there exists a positive definite matrix D such that DAD^{-1} is symmetric.

Proof: For any non-singular $N \times N$ diagonal matrix D, with the k^{th} diagonal element denoted by d_k, we have, since A is tridiagonal, that $B = DAD^{-1}$ is tridiagonal with non-zero elements given by

$$b_{k+1,k} = \frac{d_{k+1}}{d_k} a_{k+1,k}\, , \quad b_{k,k} = a_{k,k}\, , \quad b_{k,k+1} = \frac{d_k}{d_{k+1}} a_{k,k+1},$$

$$k = 1,\ldots,N-1, \qquad b_{N,N} = a_{N,N} \quad .$$

If we choose d_1 to be an arbitrary positive number and take the remaining elements of D to be

$$(4\text{-}5) \qquad d_{k+1} = \left(\frac{a_{k,k+1}}{a_{k+1,1}} \right)^{\frac{1}{2}} d_k \quad , \qquad k = 1,\ldots,N-1,$$

then B is symmetric. This completes the proof of Lemma 4-2.

4.2 Typical Behavior of the Coeffieicnts of L_h

We now examine the behavior of the coefficients of two finite difference approximations to $\mathcal{L}$, namely, the two studied in Chapter III, $L_h^{(1)}$ and $L_h^{(2)}$.

For the difference operator $L_h^{(1)}$, given by (3-8), we note that the ratio of coefficients, $\frac{\gamma_j}{\alpha_{j+1}}$, used to obtain the symmetrizing diagonal matrix D is $\frac{\gamma_j}{\alpha_{j+1}} = \frac{j^2 + \frac{1}{2}j}{j^2 + 3j/2 + \frac{1}{2}}$, $j = 1,\ldots,N-1$. As $j \to \infty$, this ratio of polynomials tends to one.

The corresponding ratio for the operator $L_h^{(2)}$, which is defined by (3-14), is $\frac{\gamma_j}{\alpha_{j+1}} = \left(\frac{j}{j+1}\right)^{2\nu+1} \frac{j + \nu + \frac{1}{2}}{j - \nu - \frac{1}{2}}$, $j = 2,\ldots, N-1$. As $j \to \infty$, this ratio also tends to one. All the operators which we consider as approximations of $\mathcal{L}$ in this chapter are such that the ratio has the form

$$\frac{\gamma_j}{\alpha_{j+1}} = \left(\frac{j}{j+1}\right)^{\ell} r(j)\ , \quad j = 3,\ldots,N-1,$$

where ℓ is some real number and $r(x)$ is a rational function of the form

$$(4\text{-}6) \qquad r(x) = \frac{(x + a_1)\ldots(x + a_s)}{(x + b_1)\ldots(x + b_s)}$$

for some positive integer s and some sets of values $\{a_k\}$,

$\{b_k\}$ such that a_k, $b_k > -2$ for $k = 1,\ldots,S$. In order that we might examine all such operators, we derive error bounds on the eigenvalue estimation for operators which satisfy such a condition.

4.3 Error Bounds for Eigenvalue Estimation

We now prove the following for operators of the form discussed in the preceding section.

Lemma 4-3: For any L_h as in (4-2) which is such that

$$\frac{\gamma_{j-1}}{\alpha_j} = \left(\frac{j}{j-1}\right)^{\ell} r(j) = \left(\frac{j}{j-1}\right)^{\ell} \prod_{k=1}^{s} \frac{(j+a_k)}{(j+b_k)} > 0, \quad j=3,\ldots,N,$$

where $a_k, b_k > -2$ and ℓ is some real number, there exist positive constants c_1 and c_2 such that

$$c_1 k^{2+m} \le d_k^2 \le c_2 k^{2+m} \quad , \quad k = 2,\ldots,N,$$

where $D = \text{diag}\,(d_1,\ldots,d_N)$ is as in (4-5) for A, the matrix associated with $\frac{1}{x_j^2} L$, and $m = \ell + \sum_{k=1}^{s} (a_k - b_k)$.

Proof: From (4-5) we have

$$d_k^2 = k^2 d_2^2 \prod_{j=3}^{k} \left(\frac{\gamma_{j-1}}{\alpha_j}\right) = k^2 d_2^2 \prod_{j=3}^{k} \left(\frac{j}{j-1}\right)^{\ell} r(j) =$$

$$= d_2^2 k^{2+\ell} 2^{-\ell} \prod_{j=3}^{k} r(j) \quad , \quad k = 2,3,\ldots,N.$$

We can choose d_1 to be any positive number and we choose it to be

$$d_1 = \left(\frac{\alpha_2}{\gamma_1}\right)^{\frac{1}{2}} 2^{\ell/2 -1}.$$

With this choice, then $d_2^2 = 2^{\ell}$ and hence

$$(4\text{-}7) \qquad d_k^2 = k^{2+\ell} \prod_{j=3}^{k} r(j), \qquad k = 2,\ldots,N.$$

By a well-known property of the gamma function ([14], p. 237), $\Gamma(x + 1) = x\Gamma(x)$, hence

$$\prod_{p=1}^{k} (p + a_i) = \Gamma(p + k + a_i) \,/\, \Gamma(a_i),$$

and thus

$$(4\text{-}8) \qquad \prod_{j=3}^{k} r(j) = \prod_{j=1}^{s} \frac{\Gamma(k + 1 + a_j)\,\Gamma(b_j)}{\Gamma(a_j)\,\Gamma(k + 1 + b_j)} =$$

$$= \prod_{\ell=1}^{s} \frac{\Gamma(b_\ell + 2)}{\Gamma(a_\ell + 2)} \prod_{j=1}^{s} \frac{\Gamma(k + 1 + a_j)}{\Gamma(k + 1 + b_j)} \,.$$

Since, by hypothesis, a_k, $b_k > -2$, the first product with index ℓ is some positive constant K.

An important fact from the theory of the gamma function, which follows from Stirling's formula ([10], p. 254-255), is

$$\frac{\Gamma(j + a)}{\Gamma(j + b)} = j^{a-b}(1+\delta_j),$$

where $\{\delta_j\}$ is a sequence of values which tends to zero as $j \to \infty$. Applying this to (4-8), we have

$$\prod_{j=3}^{k} r(j) = Kk^{a_1+\ldots+a_s-b_1-\ldots-b_s}(1 + \delta_k),$$

where $\delta_k \to 0$ as $k \to \infty$. If we combine this result with (4-7), we have

$$d_k^2 = Kk^{2+m}(1 + \delta_k) \quad , \quad k=2,\ldots,N.$$

With a choice of $c_1 = K \inf_k(1 + \delta_k)$ and $c_2 = K \sup_k(1 + \delta_k)$, the proof is complete.

<u>Lemma 4-4</u>: For $y_j = y(x_j) = J_\nu(\lambda^{\frac{1}{2}}x_j)$, $j = 1,\ldots,N$ and any real positive number λ, and $D = \operatorname{diag}(d_1,\ldots,d_n)$ such that there exists a nonnegative real number m and positive

numbers c_3 and c_4 such that

$$c_3 j^m \le d_j \le c_4 j^m \quad , \qquad j = 1, \dots, N,$$

there exists some positive constant K such that $||Dy||_2 \ge KN^{m+\frac{1}{2}}$ for $h = 1 / N+1$ sufficiently small. Furthermore, for any vector w, $||Dw||_2 \le c_4 N^{m+\frac{1}{2}} ||w||$.

<u>Proof</u>: We first examine $||Dy||_2$. We have

$$||Dy||_2^2 = \sum_{k=1}^{N} d_k^2 y_k^2 \ge \sum_{k=N/2}^{N} d_k^2 y_k^2 \ge \min_{N/2 \le k \le N} d_k^2 \sum_{k=N/2}^{N} y_k^2 \ge$$

$$\ge c_3^2 (N/2)^{2m} \sum_{k=N/2}^{N} y_k^2 \quad .$$

But we note that if we let $N \to \infty$, that is, the mesh become smaller,

$$\lim_{N \to \infty} \sum_{k=N/2}^{N} y_k^2 h = \int_{\frac{1}{2}}^{1} [y(x)]^2 dx = K_0 ,$$

and $K_0 > 0$ since, by hypothesis, y is $J_\nu(\lambda^{\frac{1}{2}}x)$ and $\lambda > 0$ so that $[y(x)]^2$ is positive except at a finite number of points in $\frac{1}{2} < x < 1$. Hence, if we choose h small enough, then $\sum_{k=N/2}^{N} y_k^2 h \ge \frac{1}{2}K_0$. If h is chosen that small, then

$$||Dy||_2^2 \geq c_3^2 (N/2)^{2m} (N+1) \sum_{k=N/2}^{N} y_k^2 h \geq c_3^2 (\tfrac{1}{2})^{2m+1} K_0 \; N^{2m+1}$$

Therefore, $||Dy||_2 \geq KN^{m+\frac{1}{2}}$ for $K = c_3 (\tfrac{1}{2})^{m+\frac{1}{2}} K_0$.

Also, we have

$$||Dw||^2 = \sum_{k=1}^{N} d_k^2 \, w_k^2 \leq \max_k \, d_k^2 \sum_{k=1}^{N} w_k^2 \leq c_4^2 \; N^{2m} \; N||w||^2 =$$

$$= c_4^2 \; N^{2m+1} \; ||w||^2.$$

This completes the proof of Lemma 4-4

<u>Theorem 4-1</u>: For any L_h as in (4-2), which has coefficients such that

$$\frac{\gamma_{j-1}}{\alpha_j} = (\frac{j}{j-1})^{\ell} \, r(j) = (\frac{j}{j-1})^{\ell} \prod_{k=1}^{s} \left(\frac{j+a_k}{j+b_k} \right), \; j=3,\ldots,N,$$

where $a_k, b_k > -2$ and ℓ is some real number, and for any eigenvalue, λ, of (4-1) and corresponding eigenfunction, $y(x)$, if $\ell + \sum_{k=1}^{s} (a_k - b_k) \geq -2$ and the vector τ is defined by

$$\tau_j \equiv \frac{1}{x_j^2} [L_h - \mathcal{L}] \; y(x_j) \qquad ,j = 1,\ldots,N,$$

then for h sufficiently small,

$$\min_k |\Lambda_k - \lambda| \leq C||\tau||,$$

where C is some positive constant and $\{\Lambda_k\}_{k=1}^N$ are the eigenvalues of the tridiagonal matrix A of coefficients of $\frac{1}{x_j^2}L_h$.

Proof: By Lemma 4-2, we know that there exists a positive diagonal matrix D which symmetrizes A. We define the vector y by

$$y = (y(x_1),\ldots,y(x_N)), \qquad y(x_j) = J_\nu(\lambda^{\frac{1}{2}}x_j).$$

Then, by Lemma 4-1,

$$\text{(4-9)} \qquad \min_k |\Lambda_k - \lambda| \le \frac{||D\tau||_2}{||Dy||_2},$$

since $\tau = (A-\lambda I)y$. By Lemma 4-3, there exist positive constants c_1 and c_2 and $m = \ell + \sum_{k=1}^{s}(a_k - b_k)$ such that

$$c_1\ k^{m+2} \le d_k^2 \le c_2\ k^{m+2} \quad ,\ k = 2,\ldots,N$$

But by hypothesis, $m + 2 \ge 0$ so that we can apply Lemma 4-4 to obtain

$$\text{(4-10)} \qquad ||Dy||_2 \ge KN^{m+\frac{1}{2}}, \quad ||D\tau||_2 \le c_2\ N^{m+\frac{1}{2}}\ ||\tau||,$$

for some constant K and h sufficiently small.

We can combine (4-9) and (4-10) to obtain

$$\min_k |\Lambda_k - \lambda| \le \frac{c_2}{K} ||\tau|| = C||\tau||.$$

This completes the proof of Theorem 4-1.

We now apply Theorem 4-1 to the operator $L_h^{(2)}$.

<u>Corollary 4-1</u>: The eigenvalues $\{\Lambda_k\}_{k=1}^N$ of the tri-diagonal matrix A of coefficients of $\frac{1}{x_j^2} L_h^{(2)}$ are such that, for any given eigenvalue λ of (4-1) and for h sufficiently small,

$$\min_k |\Lambda_k - \lambda| = O(h^2).$$

<u>Proof</u>: We have shown in Section 4.2 that for the operator $L_h^{(2)}$,

$$\frac{\gamma_{k-1}}{\alpha_k} = \left(\frac{k-1}{k}\right)^{2\nu+1} \frac{k+\nu+\frac{1}{2}}{k-\nu+\frac{1}{2}} \qquad , \ k=3,\ldots,N.$$

For this case, in Theorem 4-1 we have $\ell=-2\nu-1$ and $m=-1$. Also, from lemma 3-2

$$(\mathcal{L}-L_h^{(2)})\ u(x_j) = (c_3 x_j^{\nu+1} + x_j^{\nu+2})\ O(h^2)\ ,\ j=1,\ldots N$$

for $u \in \mathfrak{F}$ with $c_1=0$. But all eigenfunctions of (4-1) are of this form, and further, all eigenfunctions are such that $c_3=0$. Therefore,

$$|\tau_j| = |\frac{1}{x_j^2}(\mathcal{L}-L_h^{(2)})\, u(x_j)| = O(h^2) \qquad , \; j=1,\ldots,N.$$

and Theorem 4-1 implies

$$\min_k |\Lambda_k - \lambda| = O(h^2).$$

This completes the proof of Corollary 4-1.

See Example 6.2.3 in Chapter VI for numerical results of using the operator $L_h^{(2)}$ to obtain approximations of the eigenvalues of (4-1).

4.4 Analysis of the Truncation Error for a Class of Difference Operators

In this section, we study the behavior of the local truncation error of $\mathcal{L}$ and an operator $L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$, for φ some function in $C[0,1]$ with $\varphi(0)=0$ and the additional restriction that the set x^ν, $x^{\nu+2}$, $\varphi(x)$ is linearly independent on any set of three distinct points in $(0,1)$. The operator $L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$ is of the form

$$\text{(4-11)} \qquad L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}u_j = \alpha_j u_{j-1} + \beta_j u_j + \gamma_j u_{j+1}$$

for any set of real numbers $\{u_j\}$ and coefficients α_j, β_j, γ_j are determined as in Section 3.2 by the equations

$$L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}[ax_1^\nu + bx_1^{\nu+2}] = \mathcal{L}[ax_1^\nu + bx_1^{\nu+2}]$$

$$L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}[ax_j^\nu + bx_j^{\nu+2} + c\varphi(x_j)] = \mathcal{L}[ax_j^\nu + bx_j^{\nu+2} + c\varphi(x_j)],$$

$$j=2,\ldots N,$$

for all a,b,c.

<u>Theorem 4-2</u>: Let $u(x) = J_\nu(\lambda^{\frac{1}{2}}x)$ with $u(1) = 0$ denote an eigenfunction of (4-1). If φ is such that the coefficient β_j of $L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$ satisfies

$$|\beta_j - 2j^2| < K_0 \quad , \; j=2,\ldots,N,$$

for some K_0 independent of h, then for h sufficiently small,

$$(4\text{-}12) \qquad \frac{1}{x_j^2}[L_h\{x^\nu, x^{\nu+2}, \varphi(x)\} - \mathcal{L}]u(x_j) = O(h^2) \quad , \; j=1,\ldots,N.$$

Remark: The expression

$$\frac{1}{x_j^2}[L_h\{x^\nu, x^{\nu+2}, \varphi(x)\} - \mathcal{L}]u(x_j)$$

is the truncation error τ_j as in Theorem 4-1.

<u>Proof</u>: Since $L_h\{x^\nu, x^{\nu+2}, \varphi\}u_1$ is the same as $L_h^{(2)}u_1$, by Lemma 3-2

$$\frac{1}{x_j^2}(L_h\{x^\nu, x^{\nu+2}, \varphi(x)\} - \mathcal{L})u_1 = x_1^\nu O(h^2) = O(h^{2+\nu}),$$

since $u \in \mathfrak{F}$ with $c_1 = 0$. Hence (4-12) is true at j=1.

The coefficients α_j and γ_j of $L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$ can be expressed in terms of β_j as

$$
\begin{aligned}
\alpha_j &= \left(\frac{j}{j-1}\right)^\nu\left[-\frac{1+2j}{4j}\,\beta_j + (\nu+1)j\right], \qquad j = 2,\ldots,N \\
\gamma_j &= \left(\frac{j}{j+1}\right)^\nu\left[\frac{1-2j}{4j}\,\beta_j - (\nu+1)j\right], \qquad j = 2,\ldots,N.
\end{aligned}
\tag{4-13}
$$

If we use Taylor's Theorem with remainder to expand $v(x_{j+1})$ and $v(x_{j-1})$ about $x=x_j$, where $v(x) = x^{-\nu}\,u(x)$, and use those coefficients, we obtain

$$
\begin{aligned}
\tau_j &= \frac{1}{x_j^2}[L_h\{x^\nu, x^{\nu+2}, \varphi(x)\} - \mathcal{L}]\, u(x_j) \\
&= \left(1 - \frac{h^2}{2x_j^2}\,\beta_j\right)\left[u_j'' - \frac{2\nu+1}{x_j}\,u_j' + \frac{\nu(\nu+2)}{x_j^2}\,u_j\right] \\
&\quad - \frac{x_j^{\nu-2}\,h^3}{6}\left\{\left[-\frac{1}{2j}\,\beta_j + \frac{2(\nu+1)}{h}\,x_j\right] v^{(3)}(x_j) - \frac{h}{4}\beta_j v^{(4)}(\xi)\right. \\
&\quad \left. - \frac{h}{4}\left[-\frac{1}{4j}\beta_j + \frac{\nu+1}{h}\,x_j\right]\left[v^{(4)}(\eta) - v^{(4)}(\zeta)\right]\right\}, \quad j=2,\ldots,N,
\end{aligned}
\tag{4-14}
$$

where $x_{j-1} < \zeta < \xi < \eta < x_{j+1}$. We note that there exists a constant K_1 such that

$$
u_j'' - \frac{2\nu+1}{x_j}\,u_j' + \frac{\nu(\nu+2)}{x_j^2}\,u_j = x_j^{2+\nu}\sum_{k=2}^{\infty} 4c_{2k}(k^2 - k)\,x_j^{2k-4} \le x_j^{2+\nu}K_1 .
$$

The hypothesis of the theorem implies that

$$|1 - \frac{h^2}{2x_j^2}\beta_j| = \frac{1}{2j^2}|2j^2 - \beta_j| \le \frac{K_0}{2j^2} = \frac{K_0}{2x_j^2}h^2$$

and hence,

$$1 - \frac{h^2}{2x_j^2}\beta_j = \frac{1}{x_j^2}O(h^2).$$

These two results give that the first term on the right side of (4-14) is $O(h^2)$. The hypothesis and the fact that $v^{(3)}(x) = xO(1)$, yield that the second term on the right side of (4-14) is $O(h^2)$. This completes the proof of Theorem 4-2.

This analysis of $L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$ includes $L_h^{(2)}$ (given by (3-14)) as a special case with the choice $\varphi(x) = x^{\nu+1}$. For $L_h^{(2)}$, $\beta_j = 2j^2$.

Instead of using the class of operators $L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$ in this theorem, we could have used those operators of the form $L_h\{x^\nu, x^{\nu+1}, \varphi(x)\}$. We expect that the operators of the latter class do not approximate $\mathcal{L}$ in (4-1) as well as those of the former, since near the singularity of the differential equation, the eigenfunctions of (4-1) behave like $c_0x^\nu + c_2x^{\nu+2} + O(x^{\nu+4})$. The truncation error for operators $L_h\{x^\nu, x^{\nu+1}, \varphi(x)\}$ and $\mathcal{L}$ with respect to any eigenfunction

$u(x)$, of (4-1), can be found by the same procedure used in proving Theorem 4-2. We obtain an expression analogous to (4-14),

$$\tau_j = \frac{1}{x_j^2} \left[L_h\{x^{\nu}, x^{\nu+1}, \varphi(x)\} - \mathcal{L}\right] u(x_j)$$

$$= \left(1 - \frac{\beta_j h^2}{2x_j^2}\right) \left(u_j'' - \frac{2\nu}{x_j} u_j' + \frac{\nu^2+\nu}{x_j^2} u_j\right)$$

(4-15)
$$+ \frac{x_j^{\nu-1} h^2}{6} \left\{ (2\nu+1) v^{(3)}(x_j) + \frac{h^2}{8} \beta_j v^{(4)}(\xi) \right.$$

$$\left. + \frac{h}{8} (2\nu+1) \left[v^{(4)}(\eta) - v^{(4)}(\zeta)\right] \right\} \quad , j=2,\ldots,N$$

where all quantities are as defined for (4-14). If the assumption

$$|\beta_j - 2j^2| < K_0 \quad , j=2,\ldots,N,$$

is made as in Theorem 4-2, we find that

$$u_j'' - \frac{2\nu}{x_j} u_j' + \frac{\nu^2+\nu}{x_j^2} u_j = x_j^{\nu} O(1).$$

Hence, for j fixed, we have for the first term of (4-15),

$$\left(1 - \frac{\beta_j h^2}{2x_j^2}\right) \left(u_j'' - \frac{2\nu}{x_j} u_j'\right) + \frac{\nu^2+\nu}{x_j^2} = x_j^{\nu-2} O(h^2) \quad ,$$

$$j = 2,\ldots,N.$$

As $h \to 0$, then τ_j, j fixed, can be no better than $O(h^{\nu})$. Therefore, τ fails to be uniformly $O(h^2)$, unless $\beta_j = 2j^2$, $j = 2,\ldots,N$, in which case $\varphi(x) = x^{\nu+2}$.

If we construct the operator $L_h\{x^{\nu}, x^{\nu+2}, x^{\mu}\}$, where μ is any real number different from ν and $\nu + 2$, we find that β_j is such that there exists K_0 such that $|\beta_j - 2j^2| < K_0$, $j = 2,\ldots,N$. If we further assume that $\mu - \nu$ is an integer (positive or negative, but different from 2), then the ratio $\frac{\gamma_{j-1}}{\alpha_j}$ is of the form $\left(\frac{j}{j-1}\right)^m r(j)$, as is required in Theorem 4-1. We have not shown that in general the above rational function $r(x)$ has the conditions necessary for Theorem 4-1 to apply, or for what values of μ, if any, it does not. If $r(x)$ is as in (4-6), and $\ell = \sum_{k=1}^{s} (a_k - b_k) \geq -2$, then Theorem 4-1 applies to $L_h\{x^{\nu}, x^{\nu+2}, x^{\mu}\}$ and, in particular, $\|\tau\| = O(h^2)$.

4.5 Error Bound for a Self-Adjoint Difference Scheme

We now treat the special case of the operator $L_h\{x^{\nu}, x^{\nu+2}, \varphi(x)\}$, constructed so as to make the tridiagonal matrix of coefficients of $\frac{1}{x_j} L_h$ symmetric. We show that Theorem 4-1 is applicable to this operator and that $\|\tau\| = O(h^2)$.

First, we note that $\frac{1}{x}\mathcal{L}$ is a self-adjoint differential operator, since it can be written in the form

$$\frac{1}{x}\mathcal{L}u(x) = (-xu'(x))' + \frac{\nu^2}{x}\, u(x).$$

Hence, it is reasonable to assume that if $\varphi(x)$ is chosen so that $\frac{1}{x_j} L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$ is self-adjoint (or equivalently, so that the tridiagonal matrix representing the finite-difference operator $\frac{1}{x_j} L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$ is symmetric), then the resulting operator which we denote by $L_h^s = L_h\{x^\nu, x^{\nu+2}, \varphi(x)\}$, is a good approximation (in some sense) to $\mathcal{L}$.

We do not determine the function $\varphi(x)$ which gives L_h^s, but we construct the finite-difference scheme. The operator L_h^s is given by

$$L_h^s u_j = \alpha_j u_{j-1} + \beta_j u_j + \gamma_j u_{j+1} \quad , \qquad j = 1, \ldots, N, \tag{4-16}$$

for any set of numbers $\{u_j\}$, where the coefficients are determined by the relations

$$\beta_1 = \frac{4}{3}(\nu + 1) \quad , \qquad \gamma_1 = -(\tfrac{1}{2})^\nu\, \frac{4}{3}(\nu + 1)$$

$$\alpha_j = \frac{j}{j-1}\,\gamma_{j-1} \quad , \quad \beta_j = -\left(\frac{j-1}{j}\right)^\nu \frac{4j}{2j+1}\alpha_j + \frac{4j^2(\nu + 1)}{(2j + 1)} \quad , \tag{4-17}$$

$$\gamma_j = \left(\frac{j-1}{j+1}\right)^\nu \frac{(2j - 1)\, j}{(2j + 1)\, (j - 1)}\, \gamma_{j-1} - \frac{4j^2\, (\nu + 1)}{(2j + 1)}\left(\frac{j}{j+1}\right)^\nu,$$

$$j = 2, \ldots, N.$$

These are obtained by first computing α_j in such a way as to guarantee the matrix of coefficients of $\frac{1}{x_j} L_h\{x^{\nu}, x^{\nu+2}, \varphi(x)\}$ is symmetric and then computing β_j and γ_j from (4-13). The first-order difference equation for γ_j in (4-17) is easily solved (see [8], p. 50) to give

$$(4\text{-}18) \qquad \gamma_j = \frac{4j(\nu + 1)}{(2j + 1)\, j^{\nu}(j + 1)^{\nu}} \left[-1 - \sum_{k=2}^{j} k^{2\nu+1} \right]$$

$$j = 2, \ldots, N.$$

From (4-17) and (4-18), one can calculate

$$(4\text{-}19) \qquad \beta_j = 4(\nu+1) \left[\frac{j^{-2\nu}}{1-j^2/4} \sum_{k=1}^{j-1} k^{2\nu+1} + \frac{1}{2} \frac{j}{1+j/2} \right], \; j=2, \ldots, N.$$

An expansion of $\frac{1}{1 - j^2/4}$ and $\frac{1}{1 + j/2}$ into power series gives

$$(4\text{-}20) \qquad \beta_j = 4(\nu + 1) \left[\{ j^{-2\nu} \sum_{k=1}^{j-1} k^{2\nu+1} + \tfrac{1}{2} j \} + K_j \right].$$

where $|K_j|$ is less than some positive real number K, for all j.

The value of the quantity in the braces in (4-20) is equal to the estimate of the integral of $x^{2\nu+1}/j^{2\nu}$ from $x = 0$ to $x = j$ by means of the trapezoid rule applied at each of $j + 1$ equal subdivisions of $0 \le x \le j$. Since the graph of the integrand is concave upward, this estimate is an upper bound on the value $j^2/(2\nu + 2)$ of the integral.

The value of the sum in the braces in (4-20) is also equal to the estimate of the integral of $x^{2\nu+1}/j^{2\nu}$ from $x = \frac{1}{2}$ to $x = j - \frac{1}{2}$ by means of the midpoint rule applied at each of the j equal subdivisions of $\frac{1}{2} \leq x \leq j - \frac{1}{2}$. Because the graph of the integrand is concave upward, this estimate is a lower bound on the value of the integral. Hence, an upper bound on the value of the quantity in the braces in (4-20) is given by

$$\frac{1}{2\nu+2}\left[(j-\tfrac{1}{2})^{2\nu+2} - (\tfrac{1}{2})^{2\nu+2}\right]\frac{1}{j^{2\nu}} = \frac{j^2}{2\nu+2} - \tfrac{1}{2}j + O(1) + \tfrac{1}{2}j =$$

$$= \frac{j^2}{2\nu+2} + O(1).$$

A combination of these last two results and (4-20) shows that there are two constants K_1, K_2 independent of j, such that

$$2j^2 + K_1 \leq \beta_j \leq 2j^2 + K_2 \quad , \qquad j = 2,3,\ldots\,.$$

Therefore, Theorem 4-2 applies to L_h^s.

We further note that for L_h^s, $\dfrac{\gamma_j}{\alpha_{j-1}} = \dfrac{j-1}{j}$ for $j = 3,\ldots,N$, which implies that Theorem 4-2 applies to L_h^s. Thus, we have shown the following.

<u>Corollary 4-2</u>: The eigenvalues $\{\Lambda_k\}_{k=1}^N$ of the tridiagonal matrix A of coefficients of $\frac{1}{x_j^2} L_h^s$ are such that, for any given eigenvalue λ of (4-1) and for h sufficiently small,

$$\min_k |\Lambda_k - \lambda| = O(h^2).$$

See Chapter VI, Example 6.2.4 for some numerical results of approximating the eigenvalues of (4-1) by replacing $\mathcal{L}$ with L_h^s .

CHAPTER V
EXACT EIGENFUNCTIONS AND EIGENVALUES OF A BESSEL DIFFERENCE OPERATOR

In this chapter we again consider the Bessel difference operator $L_h^{(2)}$, which was constructed so that it and the Bessel operator

$$\mathcal{L} = -x^2 d^2/dx^2 - x d/dx + \nu^2$$

have zero truncation error for the functions $ax^{\nu} + bx^{\nu+1} + cx^{\nu+2}$, a,b,c, arbitrary. We obtain representations for the exact eigenfunctions and eigenvalues of the difference problem

(5-1a) $\quad \frac{1}{x_j^2} L_h^{(2)} = \Lambda u_j \quad , \quad j = 1,\ldots,N,$

(5-1b) $\quad u_0 = 0, \quad u_{N+1} = 0.$

In Chapter IV it was shown that as $N \to \infty$, the eigenvalues of (5-1) converge to the eigenvalues of the Bessel differential eigenvalue problem

$$\mathcal{L}y(x) = \lambda x^2 y(x) \quad , \; 0 < x < 1$$
$$y(0) = 0 \quad , \quad y(1) = 0.$$

These representations, which are valid for $0 < \nu < 1$, are obtained by the same technique used by Boyer ([3]) who treated only the case $\nu = 0$. We show that it follows from recurrence relations for the Legendre functions that solutions of (5-1) can be expanded in terms of linear combinations of certain of these functions.

We first show that the eigenvalues of (5-1) are real and in the interval $(0, 4(N+1)^2)$.

5.1 Properties of the Eigenvalues

The tridiagonal matrix A, which has eigenvalues and eigenvectors identical to the eigenvalues and eigenfunctions of (5-1), has non-zero coefficients given by

$$a_{k,k} = \beta_k / k^2h^2 \quad , \quad k = 1,\ldots,N,$$

$$a_{k,k+1} = \gamma_k / k^2h^2 \quad , \quad a_{k+1,k} = \alpha_{k+1} / (k+1)^2h^2 \ ,$$

$$k = 1,\ldots,N-1,$$

where $L_h^{(2)}$ is of the form $L_h^{(2)} u_j = \alpha_j u_{j-1} + \beta_j u_j + \gamma_j u_{j+1}$ for $j = 1,\ldots,N$, α, β, γ, given by (3-14). We now show the following.

<u>Theorem 5-1</u>: The matrix A has eigenvalues which are all real, positive and bounded above by $4(N+1)^2$.

Proof: By Lemma 4-2, A is similar to a real symmetric B, so that the eigenvalues of A are real. By Lemma 3-1, A is similar to a matrix which has all eigenvalues with positive real part. Therefore, all eigenvalues of A are real and positive.

We let C denote the matrix given by $D^{-1}AD$, where $D = \text{diag}(1, 2^{\nu}, \ldots, N^{\nu})$. This matrix C is the same matrix that was analyzed in Lemma 3-1, since it is the matrix of coefficients of $\frac{1}{x_j^2} K_h^{(2)}$, where $K_h^{(2)}$ is given by (3-16). The nonzero elements of C are

$$c_{1,1} = \frac{4}{3h^2}(\nu+1)\,, \quad c_{1,2} = -\frac{4}{3h^2}(\nu+1)$$

$$c_{j,j-1} = -\frac{j-\nu-\frac{1}{2}}{jh^2}\,, \quad c_{j,j} = \frac{2}{h^2}\,, \quad c_{j,j+1} = -\frac{j+\nu+\frac{1}{2}}{jh^2}\,, \quad j = 2, \ldots, N-1. \tag{5-2}$$

$$c_{N,N-1} = -\frac{N-\nu-\frac{1}{2}}{Nh^2}\,, \quad c_{N,N} = \frac{2}{h^2}\,.$$

If $\nu \leq \frac{1}{2}$, then each row sum, $\sum_{k=1}^{N} |c_{j,k}|$, $j = 1, \ldots, N$, of C is less than or equal to $\frac{4}{h^2} = 4(N+1)^2$, when $0 < \nu < \frac{1}{2}$. Hence, all eigenvalues of C are no larger than $4(N+1)^2$.

Now we consider $\frac{1}{2} < \nu < 1$. The Sturm sequence, $\{f_j\}_{j=0}^{N}$, for the tridiagonal matrix C is defined ([2], p. 202) by

$$f_0(x) = 1$$

$$f_1(x) = (x - c_{1,1})\ f_0(x)$$

$$f_{j+1}(x) = (x - c_{j+1,j+1})\ f_j(x) - c_{j+1,j}c_{j,j+1}\ f_{j-1}(x)\ ,$$

$$j = 1,\ldots,N-1.$$

The value $f_N(x)$ is equal to $\det\{-C + \lambda I\}$ and thus ([2], p. 203) since the elements of C are real and its eigenvalues are real , the number of sign changes in the sequence $\{f_j\}$ is equal to the number of eigenvalues of C that are larger than x. We now show that there are no sign changes in $\{f_j\}_0^N$ when $x=4(N+1)^2$, so that this is an upper bound on the spectral radius. With $x = 4(N + 1)^2$, the Sturm sequence is

$$f_0 = 1$$

$$f_1 = \frac{4}{3h^2}(2 - \nu)\ f_0$$

$$f_{j+1} = \frac{2}{h^2}\ f_j - \frac{1}{h^4}\left(1 + \frac{1/4 - \nu^2}{j^2+j}\right)\ f_{j-1}\quad ,\ j = 1,\ldots,N - 1.$$

We use an induction argument on j to prove that the above Sturm sequence does not change sign.

Note that

$$f_1 = \frac{4}{3h^2}(2 - \nu) > \frac{1}{h^2} = \frac{1}{h^2}\ f_0 .$$

For a given value of j, suppose $f_j > \frac{1}{h^2} f_{j-1}$. Then, a computation of f_{j+1} gives

$$f_{j+1} = \frac{2}{h^2} f_j - \frac{1}{h^4}\left(1 + \frac{1/4 - \nu^2}{j^2 + j}\right) f_{j-1}$$

$$> \frac{2}{h^2} f_j - \frac{1}{h^2}\left(1 + \frac{1/4 - \nu^2}{j^2 + j}\right) f_j > \frac{1}{h^2} f_j ,$$

where the first inequality follows from the induction hypothesis and the second from the fact that $\nu > \frac{1}{2}$. Therefore, we have shown by induction that

$$f_{j+1} > f_j h^{-2} \quad , \quad j = 0, \ldots, N-1.$$

Therefore, all the eigenvalues of C are smaller than $4(N + 1)^2$. Since A is similar to C, the proof of Theorem 5-1 is complete.

5.2 Representation of Exact Eigenvalues and Eigenfunctions

Let P_r^s and Q_r^s denote the associated Legendre functions of degree r and order s of the first and second kinds, respectively. It is well-known ([12], p. 165) that any linear combination,

$$(5\text{-}3) \qquad Y_r^s = \alpha P_r^s + \beta Q_r^s \ ,$$

of these two functions satisfies the difference equation

$$(5\text{-}4) \quad (s - r - 1)\, Y^s_{r+1}(x) + (2r+1)\, xY^s_r(x) - (s+r)\, Y^s_r(x) = 0,$$
$$-1 < x < 1.$$

We now show, for fixed x and s, that the general solution of the difference equation (5-4) for $r = r_0, r_0 + 1, \ldots,$ is given by (5-3).

Another well-known ([12], p. 163) recurrence relation which is satisfied by any linear combination of P^s_r and Q^s_r, Y^s_r, as in (5-3), is

$$(5\text{-}5) \quad (x^2 - 1)\,\frac{dY^s_r(x)}{dx} - (r+1)\, xY^s_r(x) + (s-r-1)\, Y^s_{r+1}(x) = 0,$$
$$-1 < x < 1.$$

It follows from (5-5), that for any fixed s and $x = x_0$, the pair $P^s_r(x_0)$ and $Q^s_r(x_0)$ are linearly independent on the set $r = r_0, r_0 + 1, \ldots,$ since if they were not, then $P_{r_0+k}(x_0) = aQ_{r_0+k}(x_0)$ for $k = 0, 1, \ldots,$ and some constant a. If this were true, then by (5-5), $\frac{dP^s_{r_0}(x_0)}{dx} = a\frac{dQ^s_{r_0}(x_0)}{dx}$. But since $P^s_{r_0}$ and $Q^s_{r_0}$ are both solutions of the same second-order linear homogeneous differential equation on $(-1,1)$, we then have that $P^s_{r_0}(x) = aQ^s_{r_0}(x)$ for all x in $(-1,1)$. But this contradicts the fact that $P^s_{r_0}$ and $Q^s_{r_0}$ are an independent set of functions with respect to x. Hence, for fixed s and x, the general solution of the second-order linear difference equation is given by (5-3).

If we set $s = -\nu$ and $r = j - \frac{1}{2}$ in (5-4) and make the substitutions

$$P^{-\nu}_{j-\frac{1}{2}}(\cos\omega) = j^{-\nu}T_j(\omega) \quad , \; Q^{-\nu}_{j-\frac{1}{2}}(\cos\omega) = j^{-\nu}R_j(\omega),$$

$$Y^{-\nu}_{j-\frac{1}{2}}(\cos\omega) = j^{-\nu}S_j(\omega)$$

in the difference equation (5-4), we obtain

$$(5\text{-}6) \quad -\left(\frac{j}{j+1}\right)^{\nu} \frac{j+\nu+\frac{1}{2}}{jh^2} S_{j+1}(\omega) + \frac{2}{h^2}\cos\omega S_j(\omega) -$$

$$-\left(\frac{j}{j-1}\right)^{\nu} \frac{j-\nu-\frac{1}{2}}{jh^2} S_{j-1}(\omega) = 0, \quad j = 1,2,\ldots,N.$$

A rearrangement of (5-6) yields that $S_j(\omega)$ is the general solution of

$$(5\text{-}7) \quad \frac{1}{x_j^2} L_h^{(2)} S_j(\omega) = \left(\frac{4}{h^2}\sin^2\frac{\omega}{2}\right) S_j(\omega) \quad , \; j = 2,\ldots,N.$$

We can determine one of the arbitrary constants in the general solution

$$\text{(5-9)}\quad \frac{4}{3h^2}(\nu + 1)\ S_1(\omega) - (\tfrac{1}{2})^{\nu}\ \frac{4}{3h^2}(\nu + 1)\ S_2(\omega) =$$

$$= \left[\frac{4}{h^2} \sin^2 \frac{\omega}{2}\right] S_1(\omega)\ .$$

Then, $\{S_j(\omega)\}_{j=o}^{N+1}$ satisfies

$$s_o(\omega) = 0$$

$$\frac{1}{x_j^2} L_h^{(2)} S_j(\omega) = \left[4(N+1)^2 \sin^2 \frac{\omega}{2}\right] S_j(\omega) \quad , \ j = 1,2,\ldots,N.$$

For any Λ in $(o,4(N + 1)^2)$, there exists an ω in (o,π) such that

$$\Lambda = 4(N + 1)^2 \sin^2 \frac{\omega}{2}\ .$$

Hence, since by Theorem 5-1, all of the eigenvalues Λ_k of the problem (5-1) lie in $(o,4(N + 1)^2)$, any eigenfunction of (5-1) can be represented by

$$\{S_j(\omega_k)\}_{j=o}^{N+1} \quad , \qquad k = 1,\ldots,N,$$

where $S_j(\omega)$ is defined by (5-8), one of the constants is determined by (5-9), the other remaining arbitrary, and

$$\omega_k = 2 \sin^{-1} \frac{\Lambda_k^{\frac{1}{2}}}{2(N + 1)}\ .$$

CHAPTER VI
NUMERICAL RESULTS

The numerical experiments conducted in relation to this study consisted of the estimation of the smallest eigenvalue and a corresponding eigenfunction to the finite-difference eigenvalue problem

(6-1a) $L_h u_j = \Lambda x_j^2 u_j$, $j = 1,\ldots,N$,

(6-1b) $u_0 = 0$, $u_{N+1} = 0$,

where $h = (N + 1)^{-1}$, for finite-difference operators L_h, which were discussed in Chapters III and IV, as approximations to the Bessel operator $\mathcal{L} = -x^2\, d^2/dx^2 - x d/dx + \nu^2$.

6.1 Rate of Convergence

For each finite-difference operator, L_h , considered, and for each value of $h^{-1} = N + 1 = 4,8,16,32,64,128$, we obtained accurate estimates of the smallest eigenvalue, Λ_h , and corresponding eigenfunction, $u^{(h)}$, normalized by

$u^{(h)}_{(N+1)/2} = 1$, of (6-1). The eigenvalue error, which is listed in Tables 1 through 4, is computed by $|\Lambda_h - \lambda|$, where λ is the square of the smallest zero of $J_\nu(x)$, which is the smallest eigenvalue of problem (4-1). The values of λ used are found in [15]. The error in the eigenfunction is taken to be

$$\left[\sum_{j=1}^{N} (cJ_\nu(\lambda^{\frac{1}{2}} x_j) - u_j^{(h)})^2 \, h \right]^{\frac{1}{2}} \quad ,$$

where c is a constant chosen so that $cJ_\nu(\lambda^{\frac{1}{2}}/2) = 1$. Then $cJ_\nu(\lambda^{\frac{1}{2}}x)$ is the normalized eigenfunction of problem (4-1). $J_\nu(x)$ was calculated by summing the first twenty terms of its power series expansion.

In order to examine the rates of convergence for the eigenvalues and eigenfunctions, the experimental rate of convergence (ERC) was computed and tabulated. This number is defined by

$$ERC_h = \log(e_{2h}/e_h)/\log 2,$$

where e_h is the error for a given mesh size h. This number is the power of h by which the errors are approaching zero, computed from the errors with mesh size h and 2h.

6.2 Experimental Results

Example 6.2.1: The operator, $L_h^{(1)}$, as defined by (3-9), is the three-point difference operator which agrees with the Bessel operator $\mathcal{L}$ at the mesh points for all quadratic polynomials. The errors in the eigenvalue and eigenvector estimation using $L_h^{(1)}$ are listed in Table 1, for the values of h listed in Section 6.1 for 8 values of ν. We see in Table 1 that the eigenvalues are apparently converging with order 2ν and the eigenfunctions with apparent order $5\nu/3$, for $0 < \nu < 1$. From the values in Table 1, the eigenvalues and eigenfunctions appear to converge with order 2 when $\nu \geq 1$.

Example 6.2.2: In Table 2 are the results of solving the difference eigenvalue problem for the finite-difference operator $L_h\{x^\nu, x^{\nu+1}, x^\mu\}$, that is, the three-point finite-difference operator which agrees with the Bessel operator, $\mathcal{L}$, when applied to functions of the form $ax^\nu + bx^{\nu+1}$ at x_1 and $ax^\nu + bx^{\nu+1} + cx^\mu$ at x_j, $j > 1$, for $\nu = 1/4, 1/2, 3/4$ and $\mu = \nu - 1, 0, \nu + 1/2, \nu + 3$ for each value of ν. We recall from Section 4.4, that the truncation error of such a scheme over a class of functions which includes the eigenfunctions of the differential problem is guaranteed by the analysis there to be no better than $O(h^\nu)$ at the first point. In Table 2, we see that the observed rate of convergence of both

Table 1. Errors and Experimental Rates of Convergence for Example 6.2.1.

	N+1	eigenvalue error	eigenvalue ERC	eigenfunction error	eigenfunction ERC
	8	9.1E-1	0.30	9.5E-2	-0.11
	16	6.4E-1	0.50	8.3E-2	0.21
ν=0.25	32	4.4E-1	0.53	6.6E-2	0.33
	64	3.1E-1	0.53	5.0E-2	0.39
	128	2.1E-1	0.53	3.7E-2	0.43
	8	6.1E-1	0.29	6.7E-2	0.02
	16	4.0E-1	0.58	5.3E-2	0.33
ν=0.33..	32	2.6E-1	0.65	3.9E-2	0.46
	64	1.6E-1	0.67	2.7E-2	0.52
	128	1.0E-1	0.67	1.8E-2	0.56
	8	2.0E-1	-0.41	3.1E-2	0.35
	16	1.3E-1	0.64	2.0E-2	0.62
ν=0.5	32	7.1E-2	0.86	1.2E-2	0.73
	64	3.7E-2	0.94	6.9E-3	0.79
	128	1.9E-2	0.97	3.9E-3	0.83
	8	9.0E-3	4.74	1.3E-2	0.81
	16	1.6E-2	-0.82	6.5E-3	0.98
ν=0.66..	32	1.1E-2	0.54	3.1E-3	1.05
	64	5.4E-3	1.00	1.5E-3	1.08
	128	2.4E-3	1.16	7.0E-4	1.10
	8	7.0E-2	2.45	8.4E-3	1.11
	16	1.0E-2	2.79	3.6E-3	1.23
ν=0.75	32	9.3E-5	6.75	1.5E-3	1.25
	64	8.1E-4	-3.12	6.3E-4	1.24
	128	4.9E-4	0.71	2.7E-4	1.24
	8	1.6E-1	2.02	4.1E-3	1.89
	16	4.0E-2	2.00	1.0E-3	1.97
ν=1.0	32	1.0E-2	2.00	2.6E-4	1.99
	64	2.5E-3	2.00	6.5E-5	2.00
	128	6.3E-4	2.00	1.6E-5	2.00

Table 1. (cont'd)

	8	2.4E-1	2.00	6.5E-3	2.10
	16	6.1E-2	2.00	1.6E-3	2.00
ν=1.5	32	1.5E-2	2.00	4.1E-4	1.98
	64	3.8E-3	2.00	1.0E-4	1.99
	128	9.4E-4	2.00	2.6E-5	1.99
	8	3.4E-1	2.00	9.7E-3	2.20
	16	8.4E-2	2.00	2.4E-3	2.04
ν=2.0	32	2.1E-2	2.00	5.9E-4	2.00
	64	5.3E-3	2.00	1.5E-4	2.00
	128	1.3E-3	2.00	3.7E-5	2.00

the eigenvalues and the eigenfunctions is considerably better than order h^{ν}, but that the rate of convergence does appear to increase as ν is increased. We note from Table 2, that the best experimental rate of convergence occurs when $\mu = \nu + 0.5$.

Example 6.2.3: In Table 3 we have the errors and experimental rates of convergence for the difference eigenvalue problem (6-1) with $L_h = L_h\{x^{\nu}, x^{\nu+2}, x^{\mu}\}$, the three-point finite difference operator which agrees with the Bessel operator, $\mathcal{L}$, when applied to functions of the form $ax^{\nu} + bx^{\nu+2}$ at the first mesh point and $ax^{\nu} + bx^{\nu+2} + cx^{\mu}$ at the remaining mesh points, for $\nu = 1/4, 1/2, 3/4$ and $\mu = \nu - 2, 0, 1, \nu + 1$, $\nu + 4$, $\nu + 6$, for each value of ν. The operator $L_h\{x^{\nu}, x^{\nu+2}, x^{\mu}\}$ was investigated in Section 4.4 as an approximation to $\mathcal{L}$, and the two operators were noted to have truncation error order h^2. In Table 3, we have apparent h^2 convergence for both the eigenvalues and eigenfunctions in all cases except $\nu = 0.5$, $\mu = 1.5$. In this case, the eigenfunctions of the differential problem are identical to those of the difference problem on the mesh. Hence, the error given in Table 3 for this case is only the round-off error, which grows as h becomes smalleer, as one might expect. The cases in Table 3 where $\mu = \nu + 1$ represent the operator which we denote by $L_h^{(2)}$ and which is given

Table 2. Errors and Experimental Rate of Convergence for Example 6.2.2.

	N+1	eigenvalue error	eigenvalue ERC	eigenfunction error	eigenfunction ERC
	8	1.4E-1	1.46	1.6E-3	0.85
$\nu=0.25$	16	4.3E 2	1.66	1.2E-3	0.37
	32	1.3E-2	1.79	5.1E-4	1.26
$\mu=-0.75$	64	3.4E-3	1.87	1.7E-4	1.58
	128	9.1E-4	1.91	5.1E-5	1.74
	8	9.6E-2	1.80	2.5E-4	2.55
$\nu=0.25$	16	2.6E-2	1.88	2.4E-4	0.05
	32	6.8E-3	1.93	1.0E-4	1.26
$\mu=0.0$	64	1.7E-3	1.95	3.3E-5	1.60
	128	4.4E-4	1.97	9.7E-6	1.76
	8	7.2E-2	2.10	5.0E-4	0.82
$\nu=0.25$	16	1.6E-2	2.17	3.0E-4	0.72
	32	3.6E-3	2.16	1.3E-4	1.27
$\mu=0.75$	64	8.2E-4	2.12	4.2E-5	1.57
	128	1.9E-4	2.10	1.3E-5	1.73
	8	1.4E-1	1.45	1.6E-3	0.85
$\nu=0.25$	16	4.3E-2	1.66	1.2E-3	0.37
	32	1.3E-2	1.79	5.1E-4	1.25
$\mu=3.25$	64	3.4E-3	1.86	1.7E-4	1.58
	128	9.1E-4	1.91	5.1E-5	1.74
	8	4.1E-1	0.77	1.2E-2	0.14
$\nu=0.5$	16	1.7E-1	1.28	7.0E-3	0.75
	32	5.5E-2	1.63	2.7E-3	1.35
$\mu=-0.5$	64	1.6E-2	1.81	8.7E-4	1.65
	128	4.1E-3	1.91	2.5E-4	1.81
	8	1.1E-1	2.14	6.1E-4	0.00
$\nu=0.5$	16	2.5E-2	2.12	3.4E-4	0.83
	32	5.9E-3	2.07	1.2E-4	1.49
$\mu=0.0$	64	1.4E-3	2.04	3.6E-5	1.75
	128	3.6E-4	2.02	9.8E-6	1.87
	8	9.3E-2	2.31	1.1E-3	0.00
$\nu=0.5$	16	1.9E-2	2.27	6.4E-4	0.85
	32	4.3E-3	2.17	2.2E-4	1.51
$\mu=1.0$	64	1.0E-3	2.10	6.6E-5	1.77
	128	2.4E-4	2.05	1.8E-5	1.89

Table 2 (cont'd)

	N+1	eigenvalue error	eigenvalue ERC	eigenfunction error	eigenfunction ERC
	8	1.8E-1	1.59	2.2E-3	0.02
ν=0.5	16	5.3E-2	1.78	1.2E-3	0.91
	32	1.4E-2	1.89	4.0E-4	1.54
μ=3.5	64	3.7E-3	1.95	1.2E-4	1.78
	128	9.4E-4	1.97	3.2E-5	1.89
	8	1.4E-1	2.25	1.3E-3	0.59
ν=0.75	16	3.2E-2	2.14	5.9E-4	1.11
	32	7.7E-3	2.06	1.8E-4	1.69
μ=-0.25	64	1.9E-3	2.02	5.0E-5	1.88
	128	4.7E-4	2.01	1.3E-5	1.96
	8	1.4E-1	2.28	1.5E-3	0.50
ν=0.75	16	3.0E-2	2.16	6.8E-4	1.11
	32	7.2E-3	2.06	2.1E-4	1.68
μ=0.0	64	1.8E-3	2.02	5.7E-5	1.88
	128	4.4E-4	2.00	1.5E-5	1.95
	8	1.3E-1	2.32	1.6E-3	0.45
ν=0.75	16	2.9E-2	2.18	7.5E-4	1.12
	32	6.9E-3	2.07	2.3E-4	1.69
μ=1.25	64	1.7E-3	2.02	6.3E-5	1.88
	128	4.2E-4	2.01	1.6E-5	1.95
	8	2.3E-1	1.70	2.7E-3	0.46
ν=0.75	16	6.2E-2	1.87	1.1E-3	1.32
	32	1.6E-2	1.95	3.3E-4	1.74
μ=3.75	64	4.1E-3	1.98	8.8E-5	1.90
	128	1.0E-3	1.99	2.3E-5	1.96

by (3-14). We have, by Corollary 4-1, shown that its eigenvalues converge with those of the differential problem with order of convergence at least 2.

Example 6.2.4: In Table 4, we have the errors and experimental rates of convergence for the eigenvalues and eigenfunctions of the operator L_h^s, which was discussed in Section 4.5, where L_h^s is the operator of the form $L_h\{x^{\nu}, x^{\nu+2}, \varphi(x)\}$, where $\varphi(x)$ is determined so that $\frac{1}{x}L_h^s$ is self-adjoint, for $\nu = \frac{1}{4}$, 1/3,1/2,2/3,3/4. By Corollary 4-2, we have that the eigenvalues of $\frac{1}{x_j^2}L_h^s$ converge with any eigenvalue of the differential problem with order of convergence at least two. In Table 4 this convergence is observed to occur for the smallest eigenvalue λ of the differential equation problem. When $\nu=0.5$, L_h^s is identical to $L_h^{(2)}$, and hence we have the same behavior of the eigenfunction errors as noted in Example 6.2.3 with $\nu = 0.5$ and $\mu = 1.5$, that is, the eigenfunctions of the difference and differential operators are identical, and hence the errors in Table 4 are round-off errors for this case. For all values of ν in Table 4 except $\nu = 0.5$, we note that the eigenfunctions appear to be converging with order of convergence $2 + 2\nu$, a much faster rate of convergence than observed for any of the other operators which were used.

Table 3. Errors and Experimental Rates of Convergence for Example 6.2.3.

	N+1	eigenvalues error	eigenvalues ERC	eigenfunction error	eigenfunction ERC
	8	3.0E-2	2.44	7.5E-4	2.42
ν=0.25	16	6.6E-3	2.18	1.5E-4	2.28
	32	1.6E-3	2.06	3.6E-5	2.11
μ=-1.75	64	3.9E-4	2.01	8.7E-6	2.03
	128	9.8E-5	2.00	2.2E-6	2.01
	8	8.1E-2	1.93	9.8E-5	2.82
ν=0.25	16	2.0E-2	1.98	1.4E-5	2.55
	32	5.1E-3	2.00	3.0E-6	2.18
μ=0.0	64	1.3E-3	2.00	7.3E-7	2.04
	128	3.2E-4	2.00	1.8E-7	2.00
	8	8.6E-2	1.92	1.7E-4	2.94
ν=0.25	16	2.2E-2	1.98	2.8E-5	2.57
	32	5.5E-3	2.00	6.3E-6	2.16
μ=1.0	64	1.4E-3	2.00	1.5E-6	2.03
	128	3.4E-4	2.00	3.8E-7	2.00
	8	8.4E-2	1.93	1.4E-4	3.00
ν=0.25	16	2.1E-2	1.98	2.3E-5	2.62
	32	5.3E-3	2.00	5.1E-6	2.17
μ=1.25	64	1.3E-3	2.00	1.2E-6	2.03
	128	3.3E-4	2.00	3.0E-7	2.01
	8	3.4E-2	1.41	1.5E-3	3.03
ν=0.25	16	8.8E-3	1.93	3.0E-4	2.28
	32	2.2E-3	1.99	7.4E-5	2.04
μ=4.25	64	5.5E-4	2.00	1.8E-5	2.01
	128	1.4E-4	2.00	4.6E-6	2.00
	8	2.2E-1	2.20	3.6E-3	3.55
ν=0.25	16	5.4E-2	2.00	8.0E-4	2.17
	32	1.4E-2	2.00	1.9E-4	2.03
μ=6.25	64	3.4E-3	2.00	4.8E-5	2.01
	128	8.5E-4	2.00	1.2E-5	2.00

Table 3. (cont'd)

	N+1	eigenvalue error	eigenvalues ERC	eigenfunction error	eigenfunction ERC
	8	9.1E-2	2.12	4.3E-4	2.56
ν=0.5	16	2.2E-2	2.04	8.9E-5	2.28
	32	5.5E-3	2.01	2.1E-5	2.09
μ=-1.5	64	1.4E-3	2.00	5.1E-6	2.02
	128	3.4E-4	2.00	1.3E-6	2.01
	8	1.4E-1	1.96	1.0E-4	2.69
ν=0.5	16	3.4E-2	1.99	2.2E-5	2.27
	32	8.5E-3	2.00	5.2E-6	2.07
μ=0.0	64	2.1E-3	2.00	1.3E-6	2.01
	128	5.3E-4	2.00	3.2E-7	2.00
	8	1.4E-1	1.96	1.0E-4	2.79
ν=0.5	16	3.4E-2	1.99	2.1E-5	2.26
	32	8.5E-3	2.00	5.1E-6	2.05
μ=1.0	64	2.1E-3	2.00	1.3E-6	2.01
	128	5.3E-4	2.00	3.2E-7	2.00
	8	1.3E-1	1.98	3.0E-12	1.03
ν=0.5	16	3.2E-2	1.99	3.1E-12	-0.05
	32	7.9E-3	2.00	3.3E-12	-0.06
μ=1.5	64	2.0E-3	2.00	3.5E-12	-0.09
	128	5.0E-4	2.00	4.6E-12	-0.42
	8	6.7E-2	1.97	1.9E-3	3.29
ν=0.5	16	1.7E-2	2.01	4.1E-4	2.16
	32	4.1E-3	2.00	1.0E-4	2.01
μ=4.5	64	1.0E-3	2.00	2.5E-5	2.00
	128	2.6E-4	2.00	6.4E-6	2.00
	8	3.3E-1	2.51	4.0E-3	3.86
ν=0.5	16	8.1E-2	2.03	9.7E-4	2.04
	32	2.0E-2	2.00	2.4E-4	2.02
μ=6.5	64	5.0E-3	2.00	6.0E-5	2.01
	128	1.3E-3	2.00	1.5E-5	2.00

Table 3. (cont'd)

	N+1	eigenvalue error	eigenvalue ERC	eigenfunction error	eigenfunction ERC
	8	1.7E-1	2.02	1.9E-4	2.65
ν=0.75	16	4.3E-2	2.01	3.9E-5	2.25
	32	1.1E-2	2.00	9.3E-6	2.07
μ=-1.25	64	2.7E-3	2.00	2.3E-6	2.02
	128	6.8E-4	2.00	5.7E-7	2.00
	8	2.1E-1	1.96	1.3E-4	2.49
ν=0.75	16	5.2E-2	1.99	2.9E-5	2.16
	32	1.3E-2	2.00	6.9E-6	2.04
μ=0.0	64	3.3E-3	2.00	1.7E-6	2.01
	128	8.1E-4	2.00	4.3E-7	2.00
	8	2.0E-1	1.97	5.7E-5	2.29
ν=0.75	16	5.0E-2	1.99	1.3E-5	2.11
	32.	1.3E-2	2.00	3.2E-6	2.03
μ=1.00	64	3.1E-3	2.00	8.0E-7	2.01
	128	7.8E-4	2.00	2.0E-7	2.00
	8	1.7E-1	2.00	1.7E-4	3.02
ν=0.75	16	4.3E-2	2.00	3.7E-5	2.18
	32	1.1E-2	2.00	9.2E-6	2.02
μ=1.75	64	2.7E-3	2.00	2.3E-6	2.00
	128	6.8E-4	2.00	5.7E-7	2.00
	8	1.2E-1	2.33	2.4E-3	3.36
ν=0.75	16	2.8E-2	2.04	5.7E-4	2.07
	32	7.0E-3	2.01	1.4E-4	2.01
μ=4.75	64	1.8E-3	2.00	3.5E-5	2.00
	128	4.4E-4	2.00	8.8E-6	2.00
	8	4.8E-1	2.66	5.0E-3	3.96
ν=0.75	16	1.2E-1	2.04	1.2E-3	2.00
	32	2.9E-2	2.01	3.0E-4	2.02
μ=6.75	64	7.2E-3	2.00	7.6E-5	2.01
	128	1.8E-3	2.00	1.9E-5	2.00

6.3 Numerical Techniques

The technique used to estimate the smallest eigenvalue of the N $\times$ N Matrix A associated with the difference eigenvalue problem (6-1), is described in this section. We chose an initial approximation, $y_j^{(0)} = \sin\pi jh$ as an estimate of the eigenvector associated with the smallest eigenvalue. A corresponding guess for the eigenvalue, $\lambda^{(0)}$, was made by computing the Rayleigh quotient, that is,

$$\lambda^{(0)} = \frac{(Ay^{(0)}, y^{(0)})}{\left(y^{(0)}, y^{(0)}\right)} \quad .$$

An improved estimate of the eigenvector, $y^{(1)}$, was then obtained by solving the linear system $Ay^{(1)} = \lambda^{(0)}y^{(0)}$. Since A is always tridiagonal, Gauss elimination works well for this purpose, and hence was used. This estimate, $y^{(1)}$, was then used in this Rayleigh quotient to obtain an improved estimate of the eigenvalue, $\lambda^{(1)}$. The above process was repeated to find successive values of $\lambda^{(N)}$ and $y^{(n)}$ until $\frac{|\lambda^{(n)} - \lambda^{(n-1)}|}{|\lambda^{(n)}|} < 10^{-8}$. At each step of the above iteration, the eigenvector was normalized so that its middle component was unity. The above convergence criterion on the $\lambda^{(n)}$ was always satisfied for $n = 2$, when N + 1 was larger than 8. The final value $\lambda^{(n)}$ and vector

Table 4. Errors and Experimental Rates of Convergence for Example 6.2.4.

	N+1	eigenvalue error	eigenvalue ERC	eigenfunction error	eigenfuction ERC
	8	7.4E-2	1.95	1.2E-5	2.27
	16	1.9E-2	1.97	2.3E-6	2.37
ν=0.25	32	4.7E-3	1.98	4.2E-7	2.44
	64	1.2E-3	1.99	7.5E-8	2.48
	128	3.0E-4	1.99	1.3E-8	2.49
	8	9.0E-2	1.96	9.4E-6	2.05
	16	2.3E-2	1.98	1.7E-6	2.47
ν=0.33...	32	5.7E-3	1.99	2.8E-7	2.60
	64	1.4E-3	1.99	4.5E-8	2.64
	128	3.6E-4	2.00	7.1E-9	2.66
	8	1.2E-1	1.98	3.0E-12	1.03
	16	3.2E-2	1.99	3.2E-12	-0.11
ν=0.5	32	7.9E-3	2.00	3.3E-12	-0.05
	64	2.0E-3	2.00	3.4E-12	-0.02
	128	5.0E-4	2.00	4.2E-12	-0.32
	8	1.7E-1	1.98	7.1E-6	2.99
	16	4.2E-2	2.00	7.8E-7	3.18
ν=0.666..	32	1.1E-2	2.00	8.3E-8	3.24
	64	2.6E-3	2.00	8.5E-9	3.27
	128	6.6E-4	2.00	8.7E-10	3.30
	8	1.9E-1	1.98	7.6E-6	3.16
	16	4.8E-2	2.00	7.6E-7	3.32
ν=0.75	32	1.2E-2	2.00	7.6E-7	3.37
	64	3.0E-3	2.00	6.8E-9	3.41
	128	7.5E-4	2.00	6.3E-10	3.43

$y^{(n)}$ were than taken as the estimates of the smallest eigenvalue and corresponding normalized eigenvector of the matrix A.

It should be noted that no known result guarantees the convergence of the above iteration scheme for computing the eigenvalues and eigenvectors of a given matrix A, for the matrices to which we have applied it, since they are all non-symmetric. However, if the above process does converge (and it did in all cases considered), one can easily verify that the value to which it converges is indeed an eigenvalue and the vector generated a corresponding eigenvector. When inspection of the eigenvector reveals that all of its components are of the same sign, then the eigenvalue must be the smallest one of A.

The above numerical experiment was run on the CDC 6500 at Purdue University.

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